Solutionbank M5
Edexcel AS and A Level Modular Mathematics

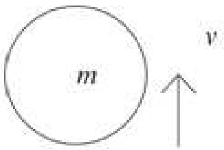
Exercise A, Question 1

Question:

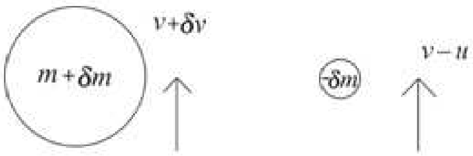
A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upwards by ejecting burnt fuel vertically downwards at a constant speed u relative to the rocket. At time t seconds after the launch the rocket has velocity v and mass $(M - kt)$. Derive the equation of motion for the rocket. Ignore air resistance.

Solution:

At time t



After an interval δt :



Change in momentum: $(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv=-mg\delta t$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

$$m=M-kt\Rightarrow\frac{\mathrm{d}m}{\mathrm{d}t}=-k,(M-kt)\frac{\mathrm{d}v}{\mathrm{d}t}-ku=-(M-kt)g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{ku}{M-kt}-g$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

A spaceship is moving in deep space with no external forces acting on it. At time t the spaceship has total mass m and is moving with velocity v. The spaceship reduces its speed by ejecting fuel from its front end with a speed c relative to itself and in the same direction as its own motion.

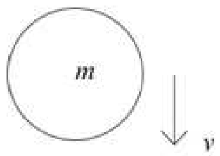
a Show that $\frac{dv}{dm} = \frac{c}{m}$.

Initially the spaceship is moving with speed V and has total mass M. Its speed is reduced to $\frac{1}{2}V$.

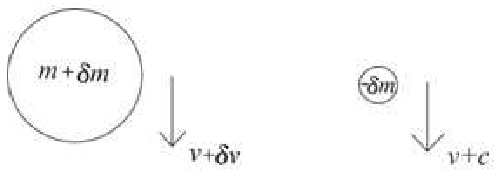
b Find the mass of fuel ejected.

Solution:

a At time t



After interval δt



$$\text{Change in momentum} \Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v+c)-mv = 0$$
$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m-c\delta m-mv = 0$$
$$m\delta v+\delta m\delta v-c\delta m = 0$$
$$\Rightarrow m\frac{\delta v}{\delta m}+\delta v-c = 0, \Rightarrow \frac{dv}{dm} = \frac{c}{m}$$

b $\frac{dv}{dm} = \frac{c}{m} \Rightarrow \int_V^{\frac{V}{2}} 1 dv = c\int_M^m \frac{1}{m} dm$

$$-\frac{V}{2} = c\left[\ln m\right]_M^m = c\ln\left(\frac{m}{M}\right)$$

$$\Rightarrow -\frac{V}{2c} = \ln\left(\frac{m}{M}\right), e^{-\frac{V}{2c}} = \frac{m}{M}, m = Me^{-\frac{V}{2c}}$$

$$\text{Mass of fuel ejected} = M\left(1-e^{-\frac{V}{2c}}\right)$$

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Exercise A, Question 3

Question:

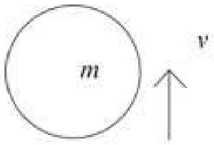
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1000 kg. The rocket burns fuel at the rate of 20 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after 30 seconds. At time t seconds $(t < 30)$ after the launch, the speed of the rocket is $v\text{ m s}^{-1}$. Air resistance may be assumed to be negligible.

a Show that $-g(50-t) = (50-t)\frac{\mathrm{d}v}{\mathrm{d}t} - 2000$.

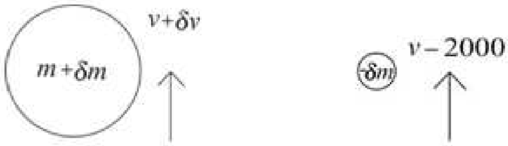
b Find the speed of the rocket when the burning stops.

Solution:

a At time t



After interval δt



Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m = 1000-20t$

$$\Rightarrow (1000-20t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times -20=-(1000-20t)g$$

Dividing by $20 \Rightarrow (50-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(50-t)$

b $\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{50-t}$

$$\Rightarrow \int_0^V 1\,\mathrm{d}v=\int_0^{30} -g+\frac{2000}{50-t}\,\mathrm{d}t$$

$$V=[-gt-2000\ln(50-t)]_0^{30}$$

$$=-30g-2000\ln 20+0+2000\ln 50\approx 1540\text{ m s}^{-1}.$$

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Exercise A, Question 4

Question:

A rocket is launched vertically upwards from rest. The rocket expels burnt fuel vertically downwards with speed u relative to the rocket. Initially the rocket has mass M. At time t the rocket has speed v and mass $M\left(1-\frac{1}{3}t\right)$. Ignore air resistance.

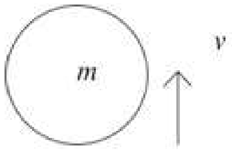
a Show that $\frac{dv}{dt}=\frac{u}{3-t}-g$.

b Find the speed of the rocket when $t=1$.

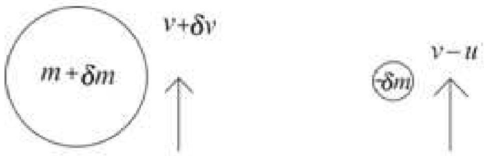
c Find the height of the rocket above the launch site when $t=1$.

Solution:

a At time t



After interval δt



$$\begin{aligned}
(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv &= -mg\delta t\\
m\delta v+\delta m\delta v+u\delta m &= -mg\delta t\\
\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} &= -mg\\
m &= M\left(1-\frac{1}{3}t\right)\\
\Rightarrow M\left(1-\frac{1}{3}t\right)\frac{\mathrm{d}v}{\mathrm{d}t}+u\left(-\frac{1}{3}M\right) &= -M\left(1-\frac{1}{3}t\right)g\\
\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\frac{1}{3}u}{1-\frac{1}{3}t}-g &= \frac{u}{3-t}-g
\end{aligned}$$

b $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{u}{3-t}-g\Rightarrow v=\int\frac{u}{3-t}-g\,\mathrm{d}t=-u\ln|3-t|-gt+C$

$t=0, v=0\Rightarrow 0=-u\ln 3+C; v=u\ln\left|\frac{3}{3-t}\right|-gt=u\ln\frac{3}{2}-g$ when $t=1$

c $v=\frac{\mathrm{d}x}{\mathrm{d}t}=u\ln\left(\frac{3}{3-t}\right)-gt, t<3$

Using integration by parts

$$\begin{aligned}
\Rightarrow x=\int u\ln\left(\frac{3}{3-t}\right)-gt\,\mathrm{d}t &= \int u\ln 3-u\ln(3-t)-gt\,\mathrm{d}t\\
&= (u\ln 3)t+u(3-t)\ln(3-t)-u(3-t)-\frac{1}{2}gt^2+C\\
t=0, x=0\Rightarrow 0 &= 0+3u\ln 3-3u+C, C=3u-3u\ln 3
\end{aligned}$$

When $t=1, x=u\ln 3+2u\ln 2-2u-\frac{g}{2}+3u-3u\ln 3=2u\ln\frac{2}{3}+u-\frac{g}{2}$

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Exercise A, Question 5

Question:

A spherical hailstone is falling under gravity in still air. At time t the hailstone has speed v. The radius r increases by condensation. Given that $\frac{dr}{dt} = kr$, where k is a constant, and neglecting air resistance,

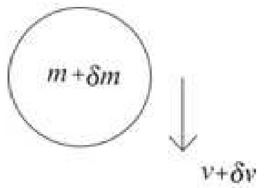
a show that $\frac{dv}{dt} = g - 3kv$,

b find the time taken for the speed of the hailstone to increase from $\frac{g}{9k}$ to $\frac{g}{6k}$.

Solution:

a At time t

After time δt :



$$\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=(m+\delta m)g\,\delta t$$

$$\Rightarrow m\frac{\delta v}{\delta t}+v\frac{\delta m}{\delta t}+\frac{\delta m\delta v}{\delta t}=mg+g\,\delta m$$

$$\text{so } m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=mg$$

The mass of the hailstone is $\lambda\times\frac{4}{3}\pi r^3$

$$\Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t}=4\lambda\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}=4\lambda\pi r^2\times kr=4k\lambda\pi r^3$$

$$\Rightarrow \lambda\times\frac{4}{3}\pi r^3\frac{\mathrm{d}v}{\mathrm{d}t}+v\times 4k\lambda\pi r^3=\lambda\times\frac{4}{3}\pi r^3 g$$

And therefore $\frac{\mathrm{d}v}{\mathrm{d}t}=g-3kv$

b $$\frac{\mathrm{d}v}{\mathrm{d}t}=g-3kv\Rightarrow t=\int_{\frac{g}{9k}}^{\frac{g}{6k}}\frac{1}{g-3kv}\mathrm{d}v=\left[-\frac{1}{3k}\ln\left|g-3kv\right|\right]_{\frac{g}{9k}}^{\frac{g}{6k}}$$

$$=-\frac{1}{3k}\ln\left(\frac{g-\frac{3kg}{6k}}{g-\frac{3kg}{9k}}\right)=-\frac{1}{3k}\ln\frac{g\times 3}{2\times 2g}=\frac{1}{3k}\ln\frac{4}{3}$$

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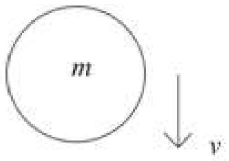
Exercise A, Question 6

Question:

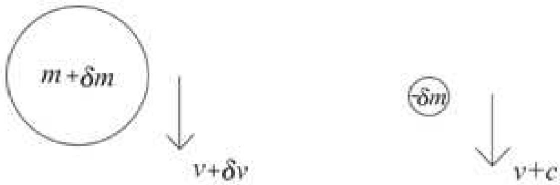
A spaceship is moving in deep space with no external forces acting on it. Initially it has total mass M and is moving with velocity V. The spaceship reduces its speed to $\frac{3}{5}V$ by ejecting fuel from its front end with a speed c relative to itself and in the same direction as its own motion. Find the mass of fuel ejected.

Solution:

At time t



After interval δt



Change in momentum $\Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v+c)-mv=0$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m-c\,\delta m-mv=0$$

$$m\delta v+\delta m\delta v-c\,\delta m=0$$

$$\Rightarrow m\frac{\delta v}{\delta m}+\delta v-c=0, \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}m}=\frac{c}{m}$$

Speed reduced from V to $\frac{3}{5}V$: $\int_V^{\frac{3V}{5}} 1\,\mathrm{d}v = c\int_M^m \frac{1}{m}\mathrm{d}m$

$$-\frac{2V}{5}=c\left[\ln m\right]_M^m = c\ln\left(\frac{m}{M}\right)$$

$$\Rightarrow -\frac{2V}{5c}=\ln\left(\frac{m}{M}\right),\ c^{-\frac{2V}{5c}}=\frac{m}{M},\ m=Me^{-\frac{2V}{5c}}$$

Mass of fuel ejected $=M\left(1-e^{-\frac{2V}{5c}}\right)$

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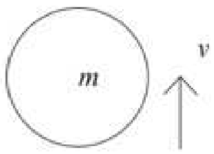
Exercise A, Question 7

Question:

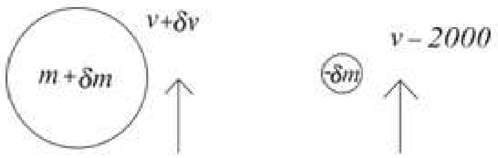
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1500 kg. The rocket burns fuel at the rate of $15\,\text{kg s}^{-1}$. The burnt fuel is ejected vertically downwards with a speed of $2000\,\text{m s}^{-1}$ relative to the rocket, and burning stops after 60 seconds. Air resistance may be assumed to be negligible.
Find the speed of the rocket when the burning stops.

Solution:

At time t



After interval δt



Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m=1500-15t$

$$\Rightarrow (1500-15t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times-15=-(1500-15t)g$$

Dividing by $15 \Rightarrow (100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(100-t)$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{100-t}$$

$$\Rightarrow \int_0^V 1\,\mathrm{d}v=\int_0^{60}-g+\frac{2000}{100-t}\,\mathrm{d}t$$

$$V=\left[-gt-2000\ln(100-t)\right]_0^{60}$$

$$=-60g-2000\ln 40+0+2000\ln 100\approx 1240\ \text{m s}^{-1}.$$

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Exercise A, Question 8

Question:

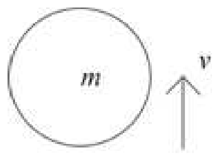
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1200 kg. The rocket burns fuel at the rate of 24 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after 30 seconds. Air resistance may be assumed to be negligible.

a Find the speed of the rocket when the burning stops.

b Find the height of the rocket above the launch pad when the burning stops.

Solution:

a At time t



After interval δt

Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m=1200-24t$

$$\Rightarrow (1200-24t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times -24=-(1200-24t)g$$

Dividing by $24 \Rightarrow (50-t)\dfrac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(50-t)$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{50-t}\Rightarrow\int_0^V 1\,\mathrm{d}v=\int_0^{30}-g+\frac{2000}{50-t}\,\mathrm{d}t$$

$$V=\left[-gt-2000\ln(50-t)\right]_0^{30}$$

$$=-30g-2000\ln 20+0+2000\ln 50\approx 1540\text{ m s}^{-1}$$

b After time t,

$$v=\left[-gt-2000\ln(50-t)\right]_0^t=-gt-2000\ln\left(\frac{50-t}{50}\right)=-gt-2000\ln\left(1-\frac{t}{50}\right)$$

so, using integration by parts,

$$x=\int_0^{30}-gt-2000\ln\left(1-\frac{t}{50}\right)\mathrm{d}t=\left[-\frac{g}{2}t^2+100\,000\left(1-\frac{t}{50}\right)\ln\left(1-\frac{t}{50}\right)-100\,000\left(1-\frac{t}{50}\right)\right]_0^{30}$$

$$=-\frac{900g}{2}+100\,000\times\frac{2}{5}\ln\frac{2}{5}-100\,000\times\frac{2}{5}+0-0+100\,000\approx 18\,900\text{ m}$$

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Exercise A, Question 9

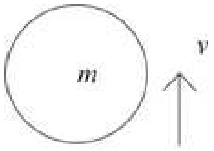
Question:

A rocket is launched vertically upwards from rest. The rocket expels burnt fuel vertically downwards with speed u relative to the rocket. Initially the rocket has mass M. At time t the rocket has speed v and mass $M\left(1-\frac{1}{4}t\right)$. Ignore air resistance.

a Find the speed of the rocket when $t = 2$.
b Find the height of the rocket above the launch site when $t = 2$.

Solution:

a At time t



After interval δt

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv=-mg\delta t$$
$$m\delta v+\delta m\delta v+u\delta m=-mg\delta t$$
$$\Rightarrow m\frac{dv}{dt}+u\frac{dm}{dt}=-mg$$
$$m=M\left(1-\frac{1}{4}t\right)$$
$$\Rightarrow M\left(1-\frac{1}{4}t\right)\frac{dv}{dt}+u\left(-\frac{1}{4}M\right)=-M\left(1-\frac{1}{4}t\right)g$$
$$\frac{dv}{dt}=\frac{\frac{1}{4}u}{1-\frac{1}{4}t}-g=\frac{u}{4-t}-g$$
$$\frac{dv}{dt}=\frac{u}{4-t}-g\Rightarrow v=\int\frac{u}{4-t}-g\,dt=-u\ln|4-t|-gt+C$$
$$t=0,\ v=0\Rightarrow 0=-u\ln 4+C$$
$$v=u\ln\left|\frac{4}{4-t}\right|-gt=u\ln\frac{4}{2}-2g=u\ln 2-2g \text{ when } t=2$$

b $v=\frac{dx}{dt}=u\ln\left(\frac{4}{4-t}\right)-gt,t<4$

Using integration by parts

$$\Rightarrow x=\int u\ln\left(\frac{4}{4-t}\right)-gt\,dt=\int u\ln 4-u\ln(4-t)-gt\,dt$$
$$=(u\ln 4)t+u(4-t)\ln(4-t)-u(4-t)-\frac{1}{2}gt^2+C$$

$t=0,x=0\Rightarrow 0=0+4u\ln 4-4u+C,C=4u-4u\ln 4$

When $t=2,x=2\times u\ln 4+2u\ln 2-2u-2g+4u-4u\ln 4=-2u\ln 2+2u-2g$

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Exercise A, Question 10

Question:

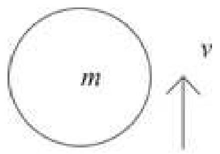
A rocket uses fuel at a rate λ kg s^{-1}. The rocket moves forwards by expelling used fuel backwards from the rocket with speed 2500 m s^{-1} relative to the rocket. At time t the rocket is moving with speed v and the combined mass of the rocket and its fuel is m. The rocket starts from rest at time $t = 0$ with a total mass 10 000 kg and reaches a final speed 5000 m s^{-1} after 200 seconds. Given that no external forces act on the rocket

a show that $m\frac{dv}{dt} = 2500\lambda$,

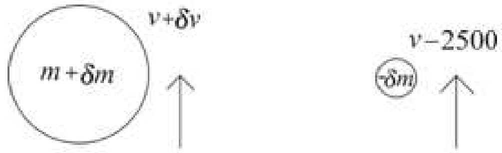
b find the value of λ kg s^{-1}.

Solution:

a At time t



After an interval δt



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2500)-mv=0$$
$$\Rightarrow m\delta v+\delta m\delta v+\delta m 2500 \ =0$$
$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2500\frac{\mathrm{d}m}{\mathrm{d}t} \ =0$$

but we are told that $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$

so $m\frac{\mathrm{d}v}{\mathrm{d}t}-2500\lambda=0, m\frac{\mathrm{d}v}{\mathrm{d}t}=2500\lambda$

b The initial mass is 10 000 and $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$, so

$(10\,000-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}=2500\lambda$

Separating the variables

$$\Rightarrow \int_0^{5000} 1\mathrm{d}v=\int_0^{200}\frac{2500\lambda}{10\,000-\lambda t}\mathrm{d}t$$
$$5000=\left[-2500\ln(10\,000-\lambda t)\right]_0^{200}=-2500\ln\left(\frac{10\,000-200\lambda}{10\,000}\right)$$
$$\Rightarrow 2=\ln\left(\frac{50}{50-\lambda}\right), e^2=\frac{50}{50-\lambda}, 50-\lambda=50e^{-2},$$
$$\lambda=50\left(1-e^{-2}\right)\approx 43.2$$

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Exercise A, Question 11

Question:

A rocket uses fuel at a rate λ. The rocket moves forwards by expelling used fuel backwards from the rocket with speed 2000 m s^{-1} relative to the rocket. At time t the rocket is moving with speed v and the combined mass of the rocket and its fuel is m. The rocket starts from rest at time $t = 0$ with a total mass 12 000 kg and reaches a speed of 5000 m s^{-1} after 3 minutes.

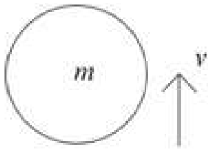
Given that no external forces act on the rocket

a show that $m\dfrac{dv}{dt} = 2000\lambda$,

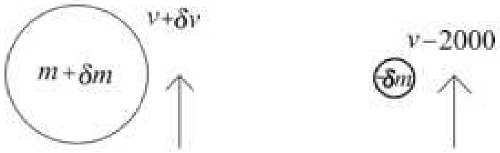
b find the greatest and the least acceleration of the vehicle during these three minutes.

Solution:

a At time t



After an interval δt



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=0$$
$$\Rightarrow m\delta v+\delta m\delta v+\delta m 2000 \ =0$$
$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t} \ =0$$

but we are told that $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$

so $m\frac{\mathrm{d}v}{\mathrm{d}t}-2000\lambda=0, m\frac{\mathrm{d}v}{\mathrm{d}t}=2000\lambda$

b The initial mass is 12 000 and $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$, so

$$(12000-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}=2000\lambda$$
$$\int_0^{5000} 1\mathrm{d}v=\int_0^{180}\frac{2000\lambda}{12\,000-\lambda t}\mathrm{d}t$$
$$5000=\left[-2000\ln(12\,000-\lambda t)\right]_0^{180}=-2000\ln\left(\frac{12\,000-180\lambda}{12\,000}\right)$$
$$\Rightarrow\frac{5}{2}=\ln\left(\frac{200}{200-3\lambda}\right), e^{\frac{5}{2}}=\frac{200}{200-3\lambda}, 200-3\lambda=200e^{-\frac{5}{2}},$$
$$\lambda=\frac{200}{3}\left(1-e^{-\frac{5}{2}}\right)\approx 61.2$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{2000\lambda}{m}\Rightarrow \text{min acceleration}=\frac{2000\times 61.2}{12\,000}=10.2\text{ m s}^{-2}$$
$$\text{max acceleration}=\frac{2000\times 61.2}{12\,000-180\times 61.2}=124\text{ m s}^{-2}$$

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Exercise A, Question 12

Question:

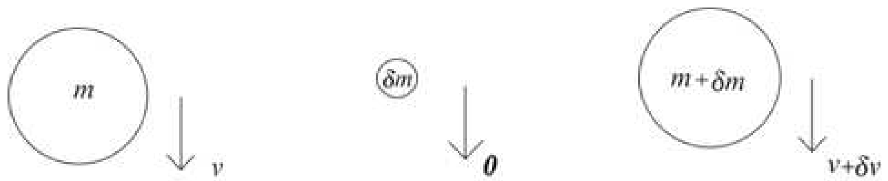
A particle falls from rest under gravity through a stationary cloud. At time t the particle has fallen a distance x, has mass m and speed v. The mass of the particle increases by accretion from the cloud at a rate of kmv, where k is a constant. Ignore air resistance. Show that

a $kv^2 = g\left(1 - e^{-2kx}\right)$,

b $x = \frac{1}{k}\ln\left[\cosh\left(\sqrt{kg}\,t\right)\right]$.

Solution:

a At time t After an interval δt :



$$[(m+\delta m)(v+\delta v)]-[mv+\delta m\times 0]=(m+\delta m)g\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=mg$$

But we are told that $\frac{\mathrm{d}m}{\mathrm{d}t}=mkv$, so $m\frac{\mathrm{d}v}{\mathrm{d}t}+v\times mkv=mg$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=g-kv^2, \Rightarrow v\frac{\mathrm{d}v}{\mathrm{d}x}=g-kv^2$$

$$\int\frac{v}{g-kv^2}\,\mathrm{d}v=\int 1\mathrm{d}x \quad \Rightarrow -\frac{1}{2k}\ln\left(g-kv^2\right)=x+C$$

$$x=0, v=0\Rightarrow -\frac{1}{2k}\ln g=C\Rightarrow x=-\frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right)$$

$$\mathrm{e}^{2kx}=\frac{g}{g-kv^2},\quad \left(g-kv^2\right)\mathrm{e}^{2kx}=g,\quad kv^2=g\left(1-\mathrm{e}^{-2kx}\right)$$

b $$v^2=\frac{g}{k}\left(1-\mathrm{e}^{-2kx}\right), v=\sqrt{\frac{g}{k}\left(1-\mathrm{e}^{-2kx}\right)}=\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\int\sqrt{\frac{g}{k}}\mathrm{d}t=\int\frac{1}{\sqrt{1-\mathrm{e}^{-2kx}}}\mathrm{d}x=\int\frac{\mathrm{e}^{kx}}{\sqrt{\mathrm{e}^{2kx}-1}}\mathrm{d}x$$

$\Rightarrow$ by using the substitution $\cosh u=\mathrm{e}^{kx}, \sinh u.\frac{\mathrm{d}u}{\mathrm{d}x}=k\mathrm{e}^{kx}$

$$\sqrt{\frac{g}{k}}\,t=\int\frac{\mathrm{e}^{kx}}{\sqrt{\mathrm{e}^{2kx}-1}}\mathrm{d}x=\frac{1}{k}\int\frac{\sinh u}{\sqrt{\cosh^2 u-1}}\mathrm{d}u=\frac{1}{k}\int 1\mathrm{d}u=\frac{u}{k}+C$$

$$t=0, x=0\Rightarrow \cosh u=1, u=\cosh^{-1}1=0, \Rightarrow C=0$$

$$\Rightarrow \sqrt{kg}\,t=u,\ \cosh(\sqrt{kg}\,t)=\mathrm{e}^{kx},\ kx=\ln[\cosh(\sqrt{kg}\,t)], x=\frac{1}{k}\ln\left[\cosh(\sqrt{kg}\,t)\right]$$

Solutionbank M5

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 13

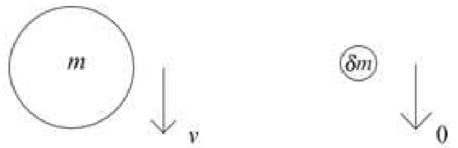
Question:

A raindrop falls through a stationary cloud. When the raindrop has fallen distance x it has mass m and speed v. The mass increases uniformly by accretion so that $m = M(1+kx)$.

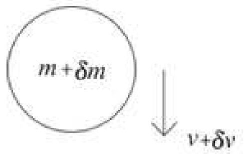
Given that $v = 0$ when $x = 0$, find an expression, in terms of M, k and x for the kinetic energy of the raindrop when it has fallen a distance x. Ignore air resistance.

Solution:

At time t



After an interval δt:



Impulse momentum: $[(m+\delta m)(v+\delta v)]-[mv]=(m+\delta m)g\delta t$

$$m\frac{\delta v}{\delta x}+v\frac{\delta m}{\delta x}+\frac{\delta m\delta v}{\delta x}=mg\frac{\delta t}{\delta x}+\delta mg\frac{\delta t}{\delta x}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}m}{\mathrm{d}x}=mg\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{mg}{v},\ mv\frac{\mathrm{d}v}{\mathrm{d}x}+v^2\frac{\mathrm{d}m}{\mathrm{d}x}=mg$$

Substituting for m:

$$M(1+kx)v\frac{\mathrm{d}v}{\mathrm{d}x}+v^2kM=M(1+kx)g$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x}+v^2\frac{k}{1+kx}=g,\ 2v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2k}{1+kx}v^2=2g$$

A linear differential equation in v^2.

Integrating factor $e^{\int\frac{2k}{1+kx}\mathrm{d}x}=e^{2\ln(1+kx)}=(1+kx)^2$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left[v^2(1+kx)^2\right]=2g(1+kx)^2,\quad v^2(1+kx)^2=\frac{2g}{3k}(1+kx)^3+C$$

$$x=0, v=0\Rightarrow 0=\frac{2g}{3k}+C,\quad v^2=\frac{2g}{3k}(1+kx)-\frac{2g}{3k(1+kx)^2},$$

$$\text{so K.E.}=\frac{1}{2}mv^2=\frac{1}{2}M(1+kx)\left[\frac{2g}{3k}(1+kx)-\frac{2g}{3k(1+kx)^2}\right]$$

$$=\frac{Mg}{3k}\left[(1+kx)^2-\frac{1}{(1+kx)}\right]$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 14

Question:

A rocket is on the ground facing vertically upwards. When launched it propels itself by ejecting mass backwards with speed u relative to the rocket at a constant rate k per unit time. The initial mass of the rocket is M. Ignore air resistance.

a Explain why it is necessary for $ku > Mg$.

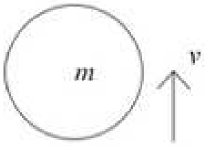
Given that $ku > Mg$,

b show that the velocity of the rocket after time t is $-u\ln\left(1-\frac{kt}{M}\right)-gt$,

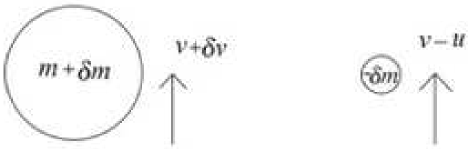
c find the height of the rocket above the ground when the mass of the rocket has reduced by one third of its initial value.

Solution:

a At time t



After an interval δt



$$\text{Change in momentum} = (m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$

$$m = M-kt \Rightarrow (M-kt)\frac{\mathrm{d}v}{\mathrm{d}t}+u(-k)=-(M-kt)g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{ku}{M-kt}-g$$

If the rocket is to be able to launch then when $t=0, \dfrac{dv}{dt}>0$

$\dfrac{ku}{M}-g>0$, i.e. $ku>Mg$

b $\dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{ku}{M-kt}-g \Rightarrow v=-u\ln(M-kt)-gt+C$

$$t=0, v=0 \Rightarrow 0=-u\ln M+C$$

$$\Rightarrow v=-u\ln(M-kt)-gt+u\ln M=-u\ln\left(\frac{M-kt}{M}\right)-gt$$

$$=-u\ln\left(1-\frac{kt}{M}\right)-gt$$

c $v=-u\ln\left(1-\dfrac{kt}{M}\right)-gt=\dfrac{\mathrm{d}x}{\mathrm{d}t}$

$$\Rightarrow x=\int -u\ln\left(1-\frac{kt}{M}\right)-gt\,\mathrm{d}t=\frac{uM}{k}\left[\left(1-\frac{kt}{M}\right)\ln\left(1-\frac{kt}{M}\right)-\left(1-\frac{kt}{M}\right)\right]-\frac{gt^2}{2}+C$$

(using integration by parts of $\ln\left(1-\dfrac{kt}{M}\right)$)

$$t = 0, x=0 \Rightarrow 0=\frac{uM}{k}\times -1+C$$

$$m=\frac{2}{3}M=M-kt, t=\frac{M}{3k}$$

$$x = \frac{uM}{k}\left[\frac{2}{3}\ln\frac{2}{3}-\frac{2}{3}\right]-\frac{gM^2}{18k^2}+\frac{uM}{k}=\frac{uM}{3k}\left[2\ln\frac{2}{3}+1-\frac{gM}{6ku}\right]$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 15

Question:

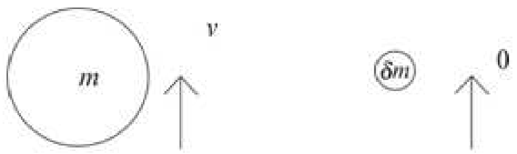
At time $t = 0$ a particle is projected vertically upwards. Initially the particle has mass M and speed gT, where T is a constant. At time t the speed of the particle is v and its mass is $Me^{\frac{t}{2T}}$. Ignore air resistance. If the added material is at rest when it is acquired, show that

a $\frac{d}{dt}\left(Mve^{\frac{t}{2T}}\right) = -Mge^{\frac{t}{2T}}$,

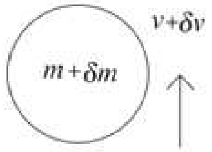
b the particle has mass $\frac{3M}{2}$ at its highest point.

Solution:

a At time t v



After an interval δt



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\,\delta t$

Taking the limit as $\delta t\to 0$

$$m\frac{dv}{dt}+v\frac{dm}{dt}=-mg,\text{ i.e. }\frac{d}{dt}(mv)=-mg$$

$$\frac{d}{dt}\left(Mve^{\frac{t}{2T}}\right)=-Me^{\frac{t}{2T}}g=-Mge^{\frac{t}{2T}}$$

b $\frac{d}{dt}\left(Mve^{\frac{t}{2T}}\right)=-Me^{\frac{t}{2T}}g,\left(Mve^{\frac{t}{2T}}\right)=\int -Mge^{\frac{t}{2T}}dt$

$$\Rightarrow Mve^{\frac{t}{2T}}=-2MgTe^{\frac{t}{2T}}+C$$

$$t=0, v=gT, C=3MgT$$

$\Rightarrow Mve^{\frac{t}{2T}}=-2MgTe^{\frac{t}{2T}}+3MgT, \Rightarrow ve^{\frac{1}{2T}}=-2gTe^{\frac{t}{2T}}+3gT$

At the highest point, $v=0$, so $0=-2gTe^{\frac{t}{2T}}+3gT, e^{\frac{t}{2T}}=\frac{3}{2}$ and

$\text{mass}=Me^{\frac{t}{2T}}=\frac{3M}{2}$

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Exercise A, Question 16

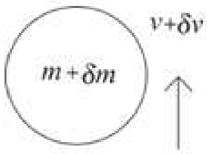
Question:

At time $t = 0$ a particle is projected vertically upwards from the ground. Initially the particle has mass M and speed $2gT$, where T is a constant. At time t the mass of the particle is $Me^{\frac{t}{T}}$. If the added material is at rest when it is acquired, show that the highest point reached by the particle is $gT^2(2 - \ln 3)$ above the ground. Ignore air resistance.

Solution:

At time t

After an interval δt



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\delta t$

Taking the limit as $\delta t\to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=-mg,\text{ i.e. }\frac{\mathrm{d}}{\mathrm{d}t}(mv)=-mg$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Mv\mathrm{e}^{\frac{t}{T}}\right)=-M\mathrm{e}^{\frac{t}{T}}g=-Mg\mathrm{e}^{\frac{t}{T}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Mv\mathrm{e}^{\frac{t}{T}}\right)=-M\mathrm{e}^{\frac{t}{T}}g,\quad\left(Mv\mathrm{e}^{\frac{t}{T}}\right)=\int -Mg\mathrm{e}^{\frac{t}{T}}dt$$

$$\Rightarrow Mv\mathrm{e}^{\frac{t}{T}}=-MgT\mathrm{e}^{\frac{t}{T}}+C$$

$$t=0, v=2gT, C=3MgT$$

$$\Rightarrow Mv\mathrm{e}^{\frac{t}{T}}=-MgT\mathrm{e}^{\frac{t}{T}}+3MgT,\Rightarrow v=-gT+3gT\mathrm{e}^{-\frac{t}{T}}$$

$$\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=-gT+3gT\mathrm{e}^{-\frac{t}{T}}, x=-gTt-3gT^2\mathrm{e}^{-\frac{t}{T}}+C$$

$$t=0, x=0,\Rightarrow C=3gT^2, x=-gTt-3gT^2\mathrm{e}^{-\frac{t}{T}}+3gT^2$$

At the highest point, $v=0,\Rightarrow \mathrm{e}^{-\frac{t}{T}}=\frac{1}{3},-\frac{t}{T}=\ln\frac{1}{3}, t=T\ln 3$

$$\Rightarrow x=-gT.T\ln 3-3gT^2.\frac{1}{3}+3gT^2$$

$$=gT^2(2-\ln 3)$$

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Exercise A, Question 17

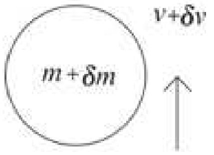
Question:

At time $t = 0$ a particle is projected vertically upwards. Initially the particle has mass M and speed gT, where T is a constant. At time t the mass of the particle is $Me^{\frac{t}{T}}$. If the added material is falling with constant speed gT when it is acquired, show that the particle has mass $\frac{3M}{2}$ at its highest point. Ignore air resistance.

Solution:

At time t

After an interval δt



Change in momentum:

$$[(m+\delta m)(v+\delta v)] - [mv - \delta m \times gT] = -(m+\delta m)g\delta t$$

$$\Rightarrow v\frac{dm}{dt} + m\frac{dv}{dt} = -mg - gT\frac{dm}{dt}$$

$$m = Me^{\frac{t}{T}} \Rightarrow \frac{dm}{dt} = \frac{M}{T}e^{\frac{t}{T}}, \frac{d}{dt}(mv) = -mg - gMe^{\frac{t}{T}}$$

$$\frac{d}{dt}\left(Me^{\frac{t}{T}}v\right) = -Me^{\frac{t}{T}}g - gMe^{\frac{t}{T}} = -2Mge^{\frac{t}{T}}$$

$$\Rightarrow Me^{\frac{t}{T}}v = \int -2gMe^{\frac{t}{T}}\,dt = -2gTMe^{\frac{t}{T}} + C$$

$$t = 0, v = gT \Rightarrow MgT = -2MgT + C, C = 3MgT$$

$$\Rightarrow Me^{\frac{t}{T}}v = -2gTMe^{\frac{t}{T}} + 3MgT$$

$$v = 0, \Rightarrow 2gTMe^{\frac{t}{T}} = 3MgT, e^{\frac{t}{T}} = \frac{3}{2}, \Rightarrow m = \frac{3M}{2}$$

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Exercise A, Question 18

Question:

A particle of mass M is projected vertically upwards in a cloud. During the motion the particle absorbs moisture from the stationary cloud so that when the particle is at distance x above the point of projection, moving with speed v, it has mass $M(1+\alpha x)$, where α is a constant. The initial speed of the particle is $\sqrt{2gk}$. Ignore air resistance.

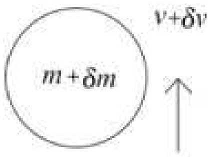
a Show that $2v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2\alpha}{1+\alpha x}v^2=-2g$.

b Show that at the greatest height, h, $(1+\alpha h)^3=1+3k\alpha$.

Solution:

a At time t

After an interval δt :



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\,\delta t$

Taking the limit as $\delta t\to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

$$m=M(1+\alpha x)\Rightarrow M(1+\alpha x)\frac{\mathrm{d}v}{\mathrm{d}t}+vM\alpha\frac{\mathrm{d}x}{\mathrm{d}t}=-M(1+\alpha x)g$$

Using $\frac{\mathrm{d}v}{\mathrm{d}t}=v\frac{\mathrm{d}v}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=v$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\alpha}{(1+\alpha x)}\frac{\mathrm{d}x}{\mathrm{d}t}=-g,\ v\frac{\mathrm{d}v}{\mathrm{d}x}+v^2\frac{\alpha}{(1+\alpha x)}=-g$$

$$\Rightarrow 2v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2\alpha}{(1+\alpha x)}v^2=-2g$$

b Multiply through the differential equation by the integrating factor (since the differential equation is linear differential equation in v^2)

$$\text{I.F.}=\mathrm{e}^{\int\frac{2\alpha}{(1+\alpha x)}\mathrm{d}x}=\mathrm{e}^{2\ln(1+\alpha x)}=(1+\alpha x)^2$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left[v^2(1+\alpha x)^2\right]=-2g(1+\alpha x)^2,\ v^2(1+\alpha x)^2=\int-2g(1+\alpha x)^2\,\mathrm{d}x$$

$$=-\frac{2g}{3\alpha}(1+\alpha x)^3+C$$

$$x=0,v=\sqrt{2gk}\Rightarrow 2gk=-\frac{2g}{3\alpha}+C,\quad C=2g\left(k+\frac{1}{3\alpha}\right)$$

At the highest point, $v=0\Rightarrow 0=-\frac{2g}{3\alpha}(1+\alpha h)^3+2g\left(k+\frac{1}{3\alpha}\right)$

$$\frac{1}{3\alpha}(1+\alpha h)^3=\left(k+\frac{1}{3\alpha}\right)$$

and therefore $(1+\alpha h)^3=1+3k\alpha$

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Exercise A, Question 19

Question:

A body of mass $3M$ contains combustible and non-combustible material in the ratio 2 : 1. The body is initially at rest and falls freely under gravity. At time t the body has speed v.

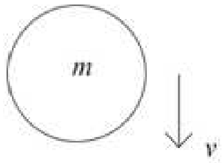
The combustible part burns at a constant rate of λM per second, where λ is a constant. The burning material is ejected vertically upwards with constant speed u relative to the body. Assuming that air resistance may be neglected,

a show that $\frac{dv}{dt} = \frac{\lambda u}{3 - \lambda t} + g$,

b find how far the body has fallen when all the combustible material has been used up.

Solution:

a At time t



After an interval δt :

$$\Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv = mg\,\delta t$$

$$m\delta v+\delta m\delta v+\delta m u = mg\,\delta t, \quad m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} = mg$$

$$m = M(3-\lambda t), \Rightarrow M(3-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}+u(-\lambda M) = M(3-\lambda t)g$$

$$(3-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}-\lambda u = (3-\lambda t)g, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda u}{3-\lambda t}+g$$

b $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda u}{3-\lambda t}+g \Rightarrow v = \int \frac{\lambda u}{3-\lambda t}+g\,\mathrm{d}t = -u\ln(3-\lambda t)+gt+C, (\lambda t < 3)$

$$t=0, v=0 \Rightarrow 0 = -u\ln 3+C$$

$$\Rightarrow v = -u\ln\left(\frac{3-\lambda t}{3}\right)+gt = -u\ln\left(1-\frac{\lambda t}{3}\right)+gt = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\Rightarrow x = \int -u\ln\left(1-\frac{\lambda t}{3}\right)+gt\,\mathrm{d}t = \frac{3u}{\lambda}\left[\left(1-\frac{\lambda t}{3}\right)\ln\left(1-\frac{\lambda t}{3}\right)-\left(1-\frac{\lambda t}{3}\right)\right]+\frac{gt^2}{2}+C$$

$$t=0, x=0 \Rightarrow 0 = -\frac{3u}{\lambda}+C$$

All combustible material used $\Rightarrow m = M(3-\lambda t) = M, \quad t = \frac{2}{\lambda}$

$$\Rightarrow x = \frac{3u}{\lambda}\left[\left(1-\frac{\lambda t}{3}\right)\ln\left(1-\frac{\lambda t}{3}\right)-\left(1-\frac{\lambda t}{3}\right)\right]+\frac{gt^2}{2}+\frac{3u}{\lambda}$$

$$= \frac{3u}{\lambda}\left[\left(1-\frac{\lambda 2}{3\lambda}\right)\ln\left(1-\frac{\lambda 2}{3\lambda}\right)-\left(1-\frac{\lambda 2}{3\lambda}\right)\right]+\frac{g4}{2\lambda^2}+\frac{3u}{\lambda}$$

$$= \frac{3u}{\lambda}\left[\frac{1}{3}\ln\frac{1}{3}-\frac{1}{3}\right]+\frac{2g}{\lambda^2}+\frac{3u}{\lambda} = \frac{3u}{\lambda}\left[\frac{1}{3}\ln\frac{1}{3}+\frac{2}{3}\right]+\frac{2g}{\lambda^2}$$

$$= \frac{u}{\lambda}\left[\ln\frac{1}{3}+2\right]+\frac{2g}{\lambda^2} = \frac{u}{\lambda}(2-\ln 3)+\frac{2g}{\lambda^2}$$

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Exercise A, Question 20

Question:

A spherical hailstone falls vertically through a stationary cloud from rest under gravity. The initial radius of the hailstone is a. As the hailstone falls its volume increases through condensation. When the radius of the hailstone is r, the rate of increase of volume is $4\pi r^2 \lambda$ and the hailstone is falling with speed v. Ignore air resistance.

a Show that, at time $t, r = a + \lambda t$.

b Show that $\frac{dv}{dt} = g - \frac{3\lambda v}{r}$.

c Find the speed of the particle when $t = \frac{a}{2\lambda}$.

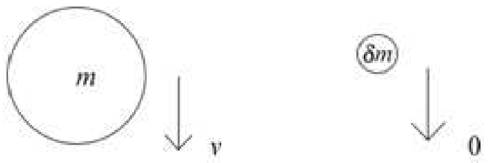
Solution:

a For the sphere, $\frac{dV}{dt} = 4\pi r^2 \lambda$, but

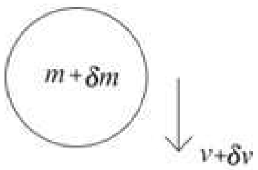
$$V = \frac{4}{3}\pi r^3 \Rightarrow 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 \lambda$$

$$\Rightarrow \frac{dr}{dt} = \lambda, r = a + \lambda t$$

b At time t



After time δt :



$$\left[(m+\delta m)(v+\delta v)\right] - \left[mv + \delta m \times 0\right] = (m+\delta m)g\,\delta t$$

$$\Rightarrow m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t} = mg + g\,\delta m \text{, so } m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

The mass of the hailstone is $\rho \times \frac{4}{3}\pi r^3$ since mass is proportional to volume

$$\Rightarrow \frac{dm}{dt} = 4\rho\pi r^2 \frac{dr}{dt} = 4\rho\pi r^2 \times \lambda$$

$$\Rightarrow \rho \times \frac{4}{3}\pi r^3 \frac{dv}{dt} + v \times 4\rho\lambda\pi r^2 = \rho \times \frac{4}{3}\pi r^3 g$$

and therefore $\frac{dv}{dt} = g - \frac{3\lambda v}{r}$

c $\frac{dv}{dt} = g - \frac{3\lambda v}{r} = g - \frac{3\lambda v}{a+\lambda t}, \frac{dv}{dt} + \frac{3\lambda v}{a+\lambda t} = g$

Using the integrating factor $e^{\int \frac{3\lambda}{a+\lambda t}dt} = e^{3\ln(a+\lambda t)} = (a+\lambda t)^3$:

$$v(a+\lambda t)^3 = \int g(a+\lambda t)^3\,dt = \frac{g}{4\lambda}(a+\lambda t)^4 + C$$

$$t = 0, v = 0, 0 = \frac{ga^4}{4\lambda} + C, \quad v(a+\lambda t)^3 = \frac{g}{4\lambda}(a+\lambda t)^4 - \frac{ga^4}{4\lambda}$$

$$v = \frac{g(a+\lambda t)}{4\lambda} - \frac{ga^4}{4\lambda(a+\lambda t)^3}$$

$$t = \frac{a}{2\lambda} \Rightarrow v = \frac{g\left(a + \lambda\frac{a}{2\lambda}\right)}{4\lambda} - \frac{ga^4}{4\lambda\left(a+\lambda\frac{a}{2\lambda}\right)^3} = \frac{3ag}{8\lambda} - \frac{2ga}{27\lambda} = \frac{65ag}{216\lambda}$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Answer this question by using calculus.
Find the moment of inertia of a thin uniform rod of mass m and length l about an axis through one end perpendicular to its length.

Solution:

Divide the rod into small pieces of length δx at a distance x from the axis.

The mass per unit length of the rod $= \frac{m}{l}$.

So the mass of a small piece $= \frac{m}{l}\delta x$.

For the whole rod

$$I = \sum_{i=1}^{n} m_i r^2$$

$$= \sum_{x=0}^{x=l} \frac{mx^2}{l}\delta x$$

As $\delta x \to 0$ summations become integrals and

$$I = \int_0^l \frac{mx^2}{l}\,\mathrm{d}x$$

$$= \left[\frac{1}{3}\frac{m}{l}x^3\right]_0^l$$

$$= \frac{1}{3}ml^2$$

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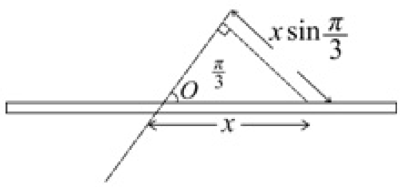
Exercise A, Question 2

Question:

Answer this question by using calculus.

Find the moment of inertia of a thin uniform rod of mass m and length l about an axis through its centre and inclined at an angle of $\frac{\pi}{3}$ to its length.

Solution:



Divide the rod into small pieces. As the mass per unit length of the rod is $\frac{m}{l}$, the mass of small piece of length δx is $\frac{m}{l}\delta x$.

If the piece is at a distance x along the rod from O, the centre of the rod, then its distance from the axis is $x\sin\frac{\pi}{3}$.

For the whole rod $I = \sum_{i=1}^{n} m_i r^2$ where $r = x\sin\frac{\pi}{3}$.

As $\delta x \to 0$, summations become integrals and

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{m}{l} x^2 \sin^2\frac{\pi}{3}\,dx \text{, where } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore\ I = \frac{m}{l} \times \frac{3}{4}\left[\frac{1}{3}x^3\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{3m}{4l}\left[\frac{l^3}{24} + \frac{l^3}{24}\right]$$

$$= \frac{ml^2}{16}$$

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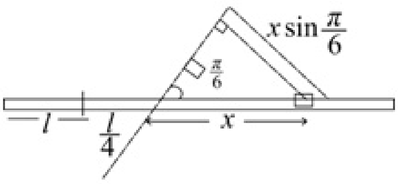
Exercise A, Question 3

Question:

Answer this question by using calculus.

Find the moment of inertia of a thin uniform rod of mass m and length $2l$ about an axis through a point at a distance $\frac{l}{4}$ from its centre and inclined at an angle of $\frac{\pi}{6}$ to its length.

Solution:



The mass per unit length of the rod $= \frac{m}{2l}$

Divide the rod into small pieces of length δx at a distance x along the rod from where the axis meets the rod.

Then a small piece is at distance $r = x\sin\frac{\pi}{6}$ from axis and the mass $m_i = \frac{m}{2l}\delta x$.

For the whole rod $I = \sum_{i=1}^{n} m_i r^2$ where $r^2 = x^2\sin^2\frac{\pi}{6} = \frac{x^2}{4}$

As $\delta x \to 0$, summations become integrals and

$$I = \int_{\frac{-5l}{4}}^{\frac{3l}{4}} \frac{m}{2l} \times \frac{x^2}{4}\,dx$$

$$= \frac{m}{8l}\left[\frac{1}{3}x^3\right]_{\frac{-5l}{4}}^{\frac{3l}{4}}$$

$$= \frac{m}{8l}\left[\frac{9l^3}{64} + \frac{125l^3}{192}\right]$$

$$= \frac{m}{8l} \times \frac{152l^3}{192}$$

$$= \frac{19ml^2}{192}$$

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Exercise A, Question 4

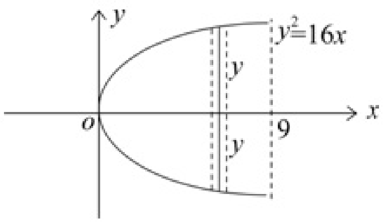
Question:

Answer this question by using calculus.

A uniform lamina, of mass M, is bounded by the curve with equation $y^2 = 16x$ and the line with equation $x = 9$.

Using calculus, find its moment of inertia about the x axis.

Solution:



Let ρ be mass per unit area of the lamina.

Then $m = \rho \int_0^9 2y \, dx$

$= 2\rho \int_0^9 4x^{\frac{1}{2}} dx$

$= 2\rho \left[\frac{2}{3} \times 4x^{\frac{3}{2}} \right]_0^9$

$= 2\rho \times \frac{2 \times 4 \times 27}{3}$

$= 144\rho$

$\therefore \rho = \frac{M}{144}$

The moment of inertia of a strip about axis through centre, perpendicular to strip $= \frac{1}{3}\delta m y^2 = \frac{1}{3}\rho 2y \cdot \delta x \cdot y^2$

So $I = \int_0^9 \frac{2}{3}\rho y^3 \, dx$

$= \frac{2}{3} \times \frac{M}{144} \int_0^9 64x^{\frac{3}{2}} dx$

$= \frac{2}{3} \times \frac{M}{144} \times 64 \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^9$

$= \frac{8}{27} M \times \frac{2}{5} \times 243$

$= \frac{144}{5} M$

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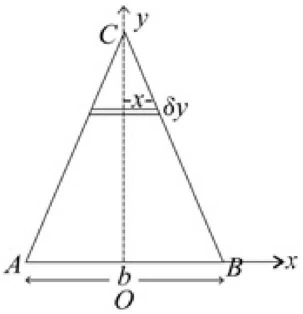
Exercise A, Question 5

Question:

Answer this question by using calculus.
Find the moment of inertia of a uniform triangular lamina of mass m which is isosceles with base b and height h about its axis of symmetry.

Solution:



Let the mass per unit area be ρ.

Then $m = \frac{1}{2} \times b \times h \times \rho$

So $\rho = \frac{2m}{bh}$

Divide the triangle into strips.
The one shown has mass $2x\delta y \times \rho$

i.e. $\frac{2m}{bh} \cdot 2x\delta y = \delta m$

The moment of inertia of the strip about the y axis is $\frac{1}{3}\delta m\, x^2 = \frac{4m}{3bh} x^3 \delta y$.

So total M.I. as $\delta y \to 0$ is given by

$$I = \int_0^h \frac{4m}{3bh} x^3 \, dy$$

The equation of the line CB is $\frac{y}{h} + \frac{2x}{b} = 1$

i.e. $x = \frac{b}{2}\left(1 - \frac{y}{h}\right)$

$$\therefore\ I = \frac{4m}{3bh} \int_0^h \frac{b^3}{8}\left(1 - \frac{y}{h}\right)^3 dy$$

$$= \frac{mb^2}{6h} \int_0^h 1 - \frac{3y}{h} + \frac{3y^2}{h^2} - \frac{y^3}{h^3} \, dy$$

$$= \frac{mb^2}{6h}\left[y - \frac{3y^2}{2h} + \frac{y^3}{h^2} - \frac{y^4}{4h^3}\right]_0^h$$

$$= \frac{mb^2}{6h} \times \frac{h}{4}$$

$$= \frac{mb^2}{24}$$

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Exercise A, Question 6

Question:

Answer this question by using calculus.

A uniform lamina, of mass M, is bounded by the positive x and y axes and the portion of the curve $y = 2\cos x$ for which $0 \le x \in \frac{\pi}{2}$.

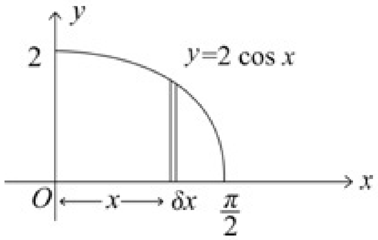
Using calculus, find its moment of inertia

a about the x axis

b about the y axis.

Solution:

a



For the strip shown

$\delta m = \rho y \delta x$

M.I. of strip about $Ox = \frac{1}{3}\delta m\, y^2$

$= \frac{1}{3}\rho y^3 \delta x$

So total $I = \frac{1}{3}\rho \int_0^{\frac{\pi}{2}} y^3 \, dx$

$= \frac{1}{3}\rho \int_0^{\frac{\pi}{2}} 2^3 \cos^3 x \, dx$

$= \frac{8}{3}\rho \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)\cos x \, dx$

$= \frac{8}{3}\rho \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}}$

$= \frac{8}{3}\rho \left[1 - \frac{1}{3}\right]$

$= \frac{16}{9}\rho$

But $M = \rho \int_0^{\frac{\pi}{2}} 2\cos x \, dx$

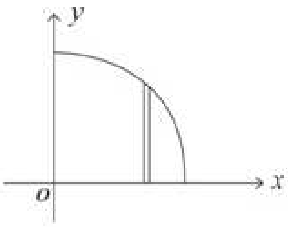
$= \rho [2\sin x]_0^{\frac{\pi}{2}}$

$= 2\rho$

$\therefore \rho = \frac{M}{2}$ and so $I = \frac{16}{9} \times \frac{M}{2}$

i.e. $I = \frac{8}{9}M$

b



M.I. of strip about the y-axis $= \delta m x^2$

$$= \rho y x^2 \delta x$$

$$= \frac{M}{2} y x^2 \delta x$$

$$\text{So } I = \frac{M}{2}\int_0^{\frac{\pi}{2}} x^2\, 2\cos x \, dx$$

$$= M\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

$$= M\left[x^2 \sin x - \int 2x \sin x \, dx\right]_0^{\frac{\pi}{2}}$$

$$= M\left[\frac{\pi^2}{4} - \left\{-2x\cos x + \int 2\cos x \, dx\right\}_0^{\frac{\pi}{2}}\right]$$

$$= M\left[\frac{\pi^2}{4} + \left[2x\cos x\right]_0^{\frac{\pi}{2}} - \left[2\sin x\right]_0^{\frac{\pi}{2}}\right]$$

$$= M\left[\frac{\pi^2}{4} - 2\right]$$

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Exercise A, Question 7

Question:

Answer this question by using additive rule and quoting known results.
A uniform ring of radius r and mass m has a particle of mass m attached to it. Find the moment of inertia of the composite body about an axis through the centre of the ring and perpendicular to the plane of the ring.

Solution:

Moment of inertia of ring $= mr^2$
Moment of inertia of particle $= mr^2$
By additive rule, moment of inertia of composite body $= 2mr^2$

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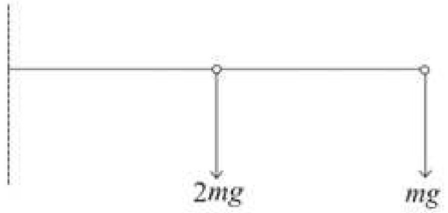
Exercise A, Question 8

Question:

Answer this question by using additive rule and quoting known results.
A uniform rod of mass $2m$ and length l has a particle of mass m fixed to one end.
Find the moment of inertia of the system about an axis through the other end of the rod and perpendicular to the rod.

Solution:



$$\text{Moment of inertia of rod } = \frac{4}{3}\cdot 2m\cdot\left(\frac{l}{2}\right)^2 = \frac{2ml^2}{3}$$

$$\text{Moment of inertia of mass } m = ml^2$$

$$\therefore \text{ Total M.I. } = \frac{2ml^2}{3} + ml^2 \text{(by additive rule)} = \frac{5ml^2}{3}$$

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Exercise A, Question 9

Question:

Answer this question by using additive rule and quoting known results.
A uniform rod of mass M and length l is attached at one of its ends to the centre of a uniform disc of radius r, which is perpendicular to the rod. Find the moment of inertia of the system about an axis along the rod.

Solution:

Moment of inertia of rod $= 0$

Moment of inertia of disc $= \frac{1}{2}mr^2$

$\therefore$ Total M. I $= \frac{1}{2}mr^2$

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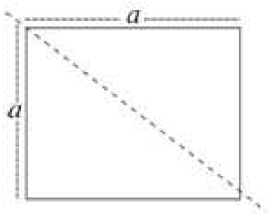
Exercise A, Question 10

Question:

Answer this question by using additive rule and quoting known results.
Four uniform rods each of mass M are rigidly jointed to form a square of side a. Find the moment of inertia of this structure about a diagonal.

Solution:



For each rod use the formula $I = \frac{4}{3}ml^2 \sin^2 \theta$, found in Example 3**c**.

$$\text{So } I = \frac{4}{3} \times M \times \left(\frac{a}{2}\right)^2 \sin^2 45°$$
$$= \frac{1}{6}Ma^2$$

But there are four rods.

$$\text{So total } I = 4 \times \frac{1}{6}Ma^2 \text{ (additive rule)}$$
$$= \frac{2}{3}Ma^2$$

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Exercise A, Question 11

Question:

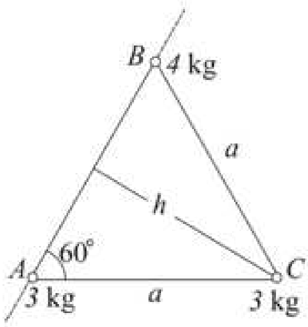
Answer this question by using additive rule and quoting known results.
Particles A, B and C of mass 3 kg, 4 kg and 3 kg respectively, are rigidly jointed by light rods to form an equilateral triangle with sides of length a.
Find the moment of inertia of the composite body about an axis

a along AB,

b through C along the axis of symmetry of the triangle which is perpendicular to AB.

Solution:

a

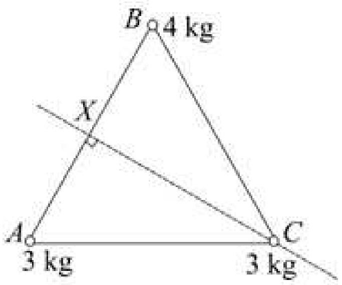


Total M.I. about $AB = 3\times 0 + 4\times 0 + 3\times h^2$ where $h = a\sin 60° = \frac{a\sqrt{3}}{2}$

$\therefore \text{M.I.}_{AB} = 3\times\frac{3a^2}{4}$

$= \frac{9a^2}{4}$

b



Let X be mid-point of AB

Total M.I. about $CX = 3\times\left(\frac{a}{2}\right)^2 + 4\times\left(\frac{a}{2}\right)^2 + 3\times 0$

$= \frac{3a^2}{4} + \frac{4a^2}{4}$

$\text{M.I.}_{cx} = \frac{7a^2}{4}$

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Exercise A, Question 12

Question:

Answer this question by using additive rule and quoting known results.
In a similar configuration to that described in question **11**, particles A, B and C of mass 3 kg, 4 kg and 3 kg respectively, are rigidly joined by **heavy** rods, each of mass 2 kg, to form an equilateral triangle with sides of length a.
Find the moment of inertia of this composite body about an axis

a along AB,

b through C along the axis of symmetry of the triangle which is perpendicular to AB.

Solution:

a M.I. of rod AB about $AB = 0$

M.I. of rod BC about $AB = \frac{4m}{3}l^2\sin^2\theta$ (from Example **3c**)

$$= \frac{4\times 2}{3}\left(\frac{a}{2}\right)^2\sin^2 60°$$

$$= \frac{2}{3}a^2\times\frac{3}{4}$$

$$= \frac{a^2}{2}$$

Similarly M.I. of rod AC about $AB = \frac{a^2}{2}$

Total M.I. of particles about $AB = \frac{9a^2}{4}$ (from Question **11**)

$\therefore$ Total M.I. about $AB = \frac{9a^2}{4}+\frac{a^2}{2}+\frac{a^2}{2}=\frac{13a^2}{4}$

b M.I. of rod AB about $CX = \frac{1}{3}\times 2\times\left(\frac{a}{2}\right)^2=\frac{a^2}{6}$

M.I. of rod BC about $CX = \frac{4\times 2}{3}\left(\frac{a}{2}\right)^2\sin^2 30° = \frac{a^2}{6}$

Similarly M.I. of rod AC about $CX = \frac{a^2}{6}$

Total M.I. of particles about $CX = \frac{7a^2}{4}$ (from Question **11**)

$\therefore$ Total M.I. about $CX = \frac{7a^2}{4}+\frac{a^2}{6}+\frac{a^2}{6}+\frac{a^2}{6}$

$$= \frac{9a^2}{4}$$

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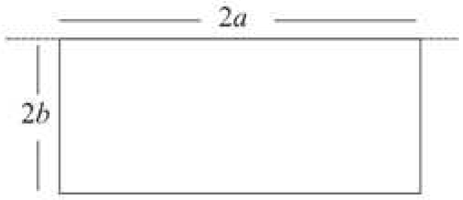
Exercise B, Question 1

Question:

Find the moment of inertia of a uniform rectangular lamina of mass m with length $2a$ and width $2b$ about an axis along the side of length $2a$.

Solution:



From the standard result $I = \frac{4}{3}mb^2$

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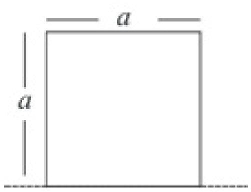
Exercise B, Question 2

Question:

Find the moment of inertia of a square lamina of mass m with sides of length a about an axis along one of the sides.

Solution:



From the standard result

$$I = \frac{4}{3}m\left(\frac{a}{2}\right)^2$$

$$= \frac{1}{3}ma^2$$

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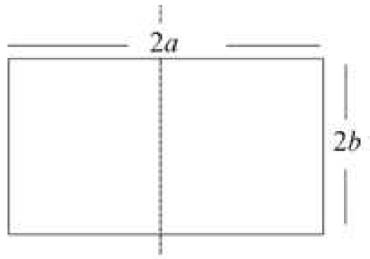
Exercise B, Question 3

Question:

Find the moment of inertia of a uniform rectangular lamina of mass m with length $2a$ and width $2b$ about an axis in the plane of the lamina, parallel to the sides of length $2b$ and bisecting the sides of length $2a$ at right angles.

Solution:



From the standard result $I = \frac{1}{3}ma^2$

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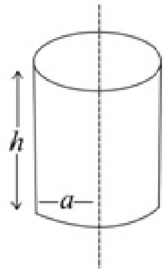
Exercise B, Question 4

Question:

Find the moment of inertia of a uniform circular solid cylinder of mass m, length h and base radius a, about its axis of symmetry.

Solution:



From standard results $I = \frac{1}{2}ma^2$

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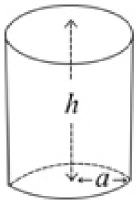
Exercise B, Question 5

Question:

Find the radius of gyration of a uniform circular hollow cylinder with height h and with a circular base of radius a of the same material, about its axis of symmetry. The total mass of the cylinder with its base is m.

Solution:



Let the mass per unit area be ρ. From standard results, the moment of inertia of the hollow cylinder is $m_1 a^2$, where m_1 is its mass and $m_1 = \rho \cdot 2\pi a h$

i.e. $I_1 = \rho \cdot 2\pi a^3 h$.

The moment of inertia of the circular base is $\frac{m_2 a^2}{2}$, where m_2 is its mass and

$m_2 = \rho \cdot \pi a^2$

i.e. $I_2 = \rho \frac{\pi a^4}{2}$

$\therefore$ Total M.I. can be obtained from additive rule and

$I = 2\pi\rho a^3 h + \frac{\pi}{2}\rho a^4 = \pi a^3 \rho (2h + \frac{a}{2})$ *

But $m = m_1 + m_2 = \rho \cdot 2\pi a h + \rho \cdot \pi a^2$

i.e. $m = \pi a \rho (2h + a) \Rightarrow \rho = \frac{m}{\pi a (2h + a)}$

Substituting into * gives

$$I = \frac{ma^2(2h + \frac{a}{2})}{2h + a}$$

i.e. $I = \frac{ma^2(4h + a)}{2(2h + a)}$

Radius of gyration, $R = \sqrt{\frac{I}{m}}$

$$= a\sqrt{\frac{4h + a}{2(2h + a)}}$$

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Exercise B, Question 6

Question:

Find the moment of inertia, about its axis of symmetry, of a uniform circular hollow cylinder of height h and base radius a, which has a circular base and circular top of twice the density of the material which forms the curved surface. The total mass of the cylinder with its base and top is m.

Solution:

Let the mass per unit area $= \rho$.
The mass of the hollow cyclinder is $2\pi ah\rho$
The mass of the circular base is $\pi a^2 \cdot 2\rho$
The mass of the circular top is $\pi a^2 \cdot 2\rho$

$\therefore$ Total mass $m = \left(2\pi ah + 2\pi a^2 + 2\pi a^2\right)\rho$ and $\rho = \dfrac{m}{2\pi ah + 4\pi a^2}$ *

The M.I. of the hollow cylinder is $(2\pi ah\rho)a^2$

M.I. of the circular base is $\left(2\pi a^2\rho\right)\dfrac{a^2}{2}$

M.I. of the circular top is $\left(2\pi a^2\rho\right)\dfrac{a^2}{2}$

$$\begin{aligned}\therefore \text{Total M.I.} &= 2\pi a^3 h\rho + \pi a^4\rho + \pi a^4\rho \\ &= \pi a^3\rho(2h + 2a)\end{aligned}$$

Substituting $\rho = \dfrac{m}{2\pi a(h+2a)}$ from *

$$\begin{aligned}\text{gives M.I.} &= \frac{m\pi a^3(2h+2a)}{2\pi a(h+2a)} \\ &= \frac{ma^2(h+a)}{h+2a}\end{aligned}$$

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Exercise B, Question 7

Question:

Use the additive rule, and the standard result for the moment of inertia of a solid sphere, to show that the radius of gyration of a uniform solid hemisphere of mass m and radius r about a diameter of the circular base is $\sqrt{\frac{2}{5}}r$.

Solution:

Let the moment of inertia of the hemisphere about a diameter of the base be I.

Then as two hemispheres form a sphere $I+I=\frac{2}{5}mr^2$, where m is mass of sphere.

So $I=\frac{1}{2}\times\frac{2}{5}mr^2$

$=\frac{2}{5}\left(\frac{m}{2}\right)r^2$

But $\frac{m}{2}=m'$ the mass of the hemisphere

So $I=\frac{2}{5}m'r^2=m'k^2$ where k, the radius of gyration, $=\sqrt{\frac{2}{5}}r$

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Exercise B, Question 8

Question:

Use the additive rule, and the standard result for the moment of inertia of a uniform circular disc, to find the radius of gyration of a uniform semicircular lamina of mass M and radius a about an axis perpendicular to the lamina through the mid-point of the straight edge.

Solution:

M.I. of circular disc about perpendicular axis through centre $= \frac{ma^2}{2}$

$\therefore$ moment of inertia, I, of semicircular disc about same axis $= \frac{ma^2}{4}$

(as $\text{I} + \text{I} = \frac{ma^2}{2}$, by additive rule)

But mass of semicircular disc $M = \frac{m}{2}$

$\therefore$ M.I. of semicircular disc $= \frac{1}{2}Ma^2$.

So radius of gyration, $k = \frac{1}{\sqrt{2}}a$.

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Exercise B, Question 9

Question:

A non-uniform solid sphere of radius R and mass M has mass kr per unit volume for all points at distance r from the centre of the sphere.

a Express k in terms of M and R.

b Use calculus to find the moment of inertia of the sphere about a diameter, giving your answer in terms of M and R.

Solution:

Divide the sphere up into concentric shells. Consider one such shell of radius r and thickness δr

its mass $\delta M \approx 4\pi r^2 \delta r \times kr$

$\therefore$ Total mass of sphere $= \sum_{r=0}^{R} 4\pi k r^3 \,\delta r$

As $\delta r \to 0$ $M = \int_0^R 4\pi k r^3 \, dr$

$$= \left[\pi k r^4\right]_0^R$$

$$\therefore \; k = \frac{M}{\pi R^4}$$

The moment of inertia of the shell

$\delta I \approx \frac{2}{3}\delta m r^2 = \frac{8}{3} k\pi r^5 \delta r$

$\therefore$ As $\delta r \to 0$ $I = \int_0^R \frac{8}{3} k\pi r^5 \, dr$

$$= \left[\frac{8}{18} k\pi r^6\right]_0^R$$

$$= \frac{4}{9}\pi R^6 k$$

$$= \frac{4}{9}\pi R^6 \times \frac{M}{\pi R^4}$$

$$= \frac{4}{9} MR^2$$

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Exercise B, Question 10

Question:

Using the formula for the moment of inertia of a uniform solid sphere,

a find the moment of inertia of a uniform spherical shell of inner radius r and outer radius R and mass m.

b Show that as $r \to R$ the moment of inertia reaches the value $\frac{2}{3}mr^2$.

Solution:

a By additive rule

Moment of inertia of sphere with radius r + Moment of inertia of shell = Moment of inertia of sphere with radius R

Let the sphere have mass per unit volume ρ.

Then moment of inertia of large sphere $= \frac{2}{5} \times \frac{4}{3}\pi R^3 \rho \times R^2$

$$= \frac{8}{15}\pi\rho R^5$$

Also moment of inertia of small sphere $= \frac{8}{15}\pi\rho r^5$

$\therefore$ Moment of inertia of shell $= \frac{8}{15}\pi\rho R^5 - \frac{8}{15}\pi\rho r^5$

i.e. $I = \frac{8}{15}\pi\rho\left(R^5 - r^5\right)$ ①

But mass of shell $= m = \left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho$

$$= \frac{4}{3}\pi\rho\left(R^3 - r^3\right) \qquad ②$$

Dividing ① by ② gives

$$\frac{I}{m} = \frac{\frac{8}{15}\pi\rho\left(R^5 - r^5\right)}{\frac{4}{3}\pi\rho\left(R^3 - r^3\right)}$$

$$= \frac{2}{5}\frac{(R-r)\left(R^4 + R^3 r + R^2 r^2 + R r^3 + r^4\right)}{(R-r)\left(R^2 + Rr + r^2\right)}$$

$$\therefore I = m \times 2\frac{\left(R^4 + R^3 r + R^2 r^2 + R r^3 + r^4\right)}{5\left(R^2 + Rr + r^2\right)}$$

b As $r \to R$

$$I = m \times \frac{2}{5} \times \frac{5R^4}{3R^2}$$

$$= \frac{2}{3}mR^2$$

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Exercise B, Question 11

Question:

Using the formula for the moment of inertia of a uniform solid cone, (found in Example 13)

a find the moment of inertia of a conical shell, with inner radius r and inner height h' and outer radius R and outer height h and mass m. You should assume that the inner and outer cone are geometrically similar.

b Show that as $r \to R$ the moment of inertia reaches the value $\frac{1}{2}mr^2$.

c Explain how you could have deduced the value of the moment of inertia by considering a circular disc divided into a large number of concentric hoops.

Solution:

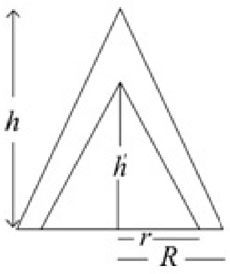
a Moment of inertia of conical shell = Moment of inertia of large cone − Moment of inertia of small cone

$$= \frac{3}{10}M_1R^2 - \frac{3}{10}M_2r^2$$

where M_1 is the mass of the large cone and M_2 is the mass of the small cone.

Let ρ be the mass per unit volume of the cones.

Then $M_1 = \frac{1}{3}\pi R^2 h\rho$ and $M_2 = \frac{1}{3}\pi r^2 h'\rho$



From similar triangles $\frac{h'}{r} = \frac{h}{R}$

So $h' = h\frac{r}{R}$

So $m = M_1 - M_2$

i.e. $m = \frac{1}{3}\pi\rho\left(R^2h - \frac{hr^3}{R}\right)$ ①

Also $I = \frac{3}{10} \times \frac{1}{3}\pi R^2 h\rho \times R^2 - \frac{3}{10} \times \frac{1}{3}\pi r^2 \frac{hr}{R}\rho \times r^2$

$$I = \frac{1}{10}\pi h\rho\left(R^4 - \frac{r^5}{R}\right) \quad ②$$

Divide equation ② by equation ① to give

$$\frac{I}{m} = \frac{\frac{1}{10}\pi h\rho\left(R^4 - \frac{r^5}{R}\right)}{\frac{1}{3}\pi\rho h\left(R^2 - \frac{r^3}{R}\right)}$$

$$= \frac{3}{10}\frac{\left(R^5 - r^5\right)}{\left(R^3 - r^3\right)}$$

i.e. $I = \frac{3m}{10}\frac{\left(R^4 + R^3r + R^2r^2 + Rr^3 + r^4\right)}{\left(R^2 + Rr + r^2\right)}$

b As $r \to R$ $I = \frac{3}{10}m \times \frac{5R^4}{3R^2} = \frac{1}{2}mR^2$

c Consider the cone divided up into a large number of thin hoops centred on its axis of symmetry.

This is similar to a disc of the same radius divided up into a large number of thin hoops.

They have the same mass distribution and so the same moment of inertia i.e. $\frac{1}{2}mr^2$.

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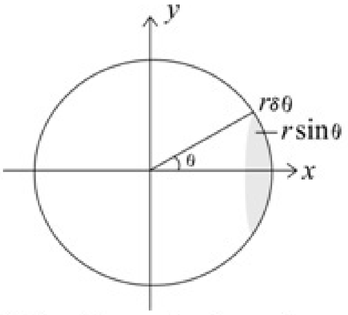
Exercise B, Question 12

Question:

Find, by integration, the moment of inertia of a uniform hollow sphere of mass m and radius r about an axis through the centre of the sphere.

Divide the sphere into composite hoops of surface area $2\pi r\sin\theta \times r\delta\theta$, where θ is the angle between the axis and the radius which joins a point on the outer circular boundary of the hoop to the centre of the sphere.

Solution:



Divide the sphere into hoops one of which is shown.

The surface area of the hoop $= 2\pi r\sin\theta \times r\delta\theta$

The mass per unit area of the sphere $= \dfrac{m}{4\pi r^2}$

$\therefore$ The mass of the hoop shown $= \dfrac{2\pi r^2 \sin\theta m\delta\theta}{4\pi r^2}$

$$= \frac{1}{2}m\sin\theta\delta\theta$$

M.I. of hoop about x-axis $=$ mass $\times$ radius^2

$$= \frac{1}{2}m\sin\theta \times r^2\sin^2\theta\delta\theta$$

Adding the moments of inertia of all such hoops and letting $\delta\theta \to 0$

$$\text{M.I. of sphere about } x\text{-axis} = \int_0^\pi \frac{mr^2}{2}\sin\theta\cdot\sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{mr^2}{2}\int_0^\pi (1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta$$

$$= \frac{mr^2}{2}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^\pi$$

$$= \frac{mr^2}{2}\left[1-\frac{1}{3}+1-\frac{1}{3}\right]$$

$$= \frac{mr^2}{2}\times\frac{4}{3}$$

$$= \frac{2mr^2}{3}$$

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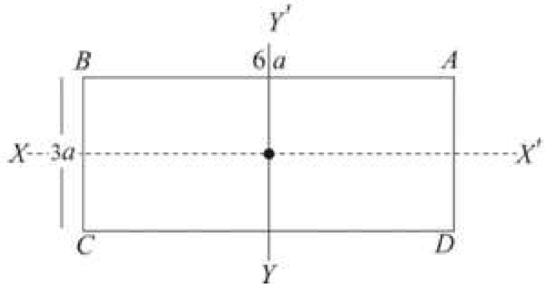
Exercise C, Question 1

Question:

A uniform lamina of mass m is in the shape of a rectangle $ABCD$ where $AB = 6a$ and $BC = 3a$. Find the moment of inertia of the lamina about an axis perpendicular to the lamina, acting through the centre of the lamina.

Solution:



Moment of inertia about $XX' = \frac{1}{3}m\left(\frac{3a}{2}\right)^2$

Moment of inertia about $YY' = \frac{1}{3}m(3a)^2$

$\therefore$ Moment of inertia about an axis perpendicular to the lamina acting through the centre of the lamina $= \frac{1}{3}m\left(\frac{3a}{2}\right)^2 + \frac{1}{3}m(3a)^2$ (using the perpendicular axes theorem).

i.e. required M.I. $= \frac{1}{3}m \times \frac{9a^2}{4} + \frac{1}{3}m \times 9a^2$

$= \frac{15ma^2}{4}$

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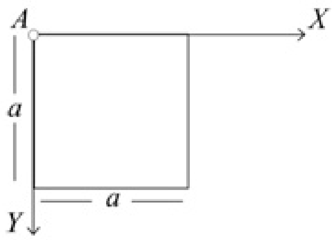
Exercise C, Question 2

Question:

Find the moment of inertia of a square lamina of mass m and side a about an axis through one corner perpendicular to the plane of the lamina.

Solution:



Choose one of the corners, A for example.

Moment of inertia of square about axis AX shown $= \frac{4}{3}m\left(\frac{a}{2}\right)^2$

$= \frac{1}{3}ma^2$

Moment of inertia of square about axis AY shown, $= \frac{4}{3}m\left(\frac{a}{2}\right)^2$ also

$= \frac{1}{3}ma^2$

$\therefore$ Moment of inertia about an axis through one corner, perpendicular to the plane is I where

$I = \frac{1}{3}ma^2 + \frac{1}{3}ma^2$ (perpendicular axes theorem)

i.e. $I = \frac{2}{3}ma^2$

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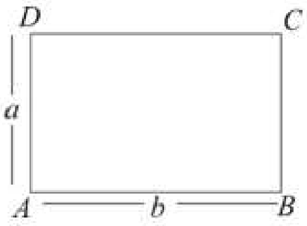
Exercise C, Question 3

Question:

Find the moment of inertia of a rectangular lamina of mass m and sides a and b about an axis through one corner perpendicular to the plane of the lamina.

Solution:



$$I_{AB} = \frac{4}{3}m\left(\frac{a}{2}\right)^2 = \frac{1}{3}ma^2$$

$$I_{AD} = \frac{4}{3}m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2$$

$\therefore$ Moment of inertia about an axis through A perpendicular to the

$$\text{lamina} = \frac{1}{3}ma^2 + \frac{1}{3}mb^2$$

$$= \frac{1}{3}m\left(a^2 + b^2\right)$$

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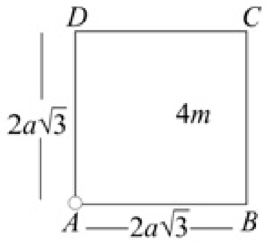
Exercise C, Question 4

Question:

A uniform square lamina $ABCD$ is of mass $4m$ and side $2a\sqrt{3}$. The axis L is a smooth fixed axis which passes through A and is perpendicular to the lamina. Show that the moment of inertia of the lamina about L is $32ma^2$.

Solution:



$$I_{AB} = \frac{4}{3} \times 4m\left(a\sqrt{3}\right)^2 = 16ma^2$$

$$I_{AD} = \frac{4}{3} \times 4m\left(a\sqrt{3}\right)^2 = 16ma^2$$

$$\therefore\ I_L = 16ma^2 + 16ma^2 \text{ (perpendicular axes theorem)}$$

$$= 32ma^2$$

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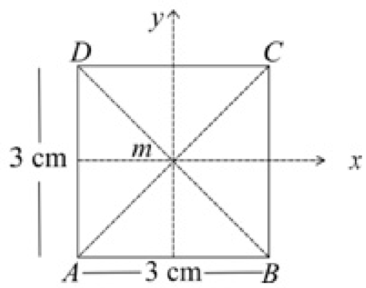
Exercise C, Question 5

Question:

A uniform lamina of mass m is in the shape of a square $ABCD$ with sides of length 3 cm. Find the moment of inertia of the lamina about the diagonal AC.

Solution:



Let centre of square be O and take x, y and z axes such that Ox is parallel to AB, Oy is parallel to AD and Oz is perpendicular to the lamina.

Then

$$I_{Ox} = \frac{1}{3}m\left(\frac{3}{2}\right)^2 = \frac{3m}{4}$$

$$I_{Oy} = \frac{1}{3}m\left(\frac{3}{2}\right)^2 = \frac{3m}{4}$$

$$\therefore\ I_{Oz} = \frac{3m}{4} + \frac{3m}{4}, \text{ by perpendicular axes theorem.}$$

$$= \frac{3m}{2}$$

Let $I_{AC} = I_{BD} = I$

Then $I + I = \frac{3m}{2}$, by perpendicular axes theorem

$\therefore I = \frac{3m}{4}$

So the moment of inertia about the diagonal $AC = \frac{3m}{4}$

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Exercise C, Question 6

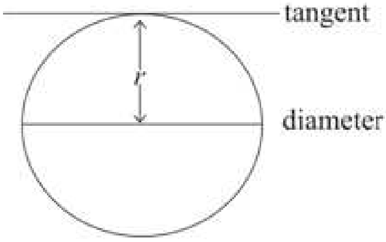
Question:

Find the radius of gyration of a uniform circular disc of radius r about a line in the plane of the disc which is tangential to the disc.

Solution:

The M.I. of the disc about an axis, through its centre, perpendicular to the disc $= \frac{mr^2}{2}$,
where m is its mass.
$\therefore$ By perpendicular axes theorem, the moment of inertia of the disc about a diameter is I where $I + I = \frac{mr^2}{2}$

$\therefore I = \frac{mr^2}{4}$



Let the moment of inertia of the disc about a tangent be I'
Then

$I' = I + mr^2$, by the parallel axes theorem

$\therefore I' = \frac{mr^2}{4} + mr^2$

$= \frac{5mr^2}{4}$

So if the radius of gyration is k

$mk^2 = 5\frac{mr^2}{4}$

$\therefore k = \frac{\sqrt{5}r}{2}$

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Exercise C, Question 7

Question:

Find the radius of gyration of a circular ring of radius r about a line in the plane of the ring which is tangential to the ring.

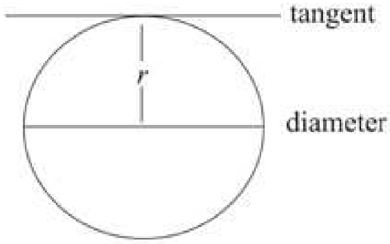
Solution:

The moment of inertia of a ring about an axis through its centre, perpendicular to the plane of the ring $= mr^2$, where m is its mass.

The moment of inertia of the ring about a diameter is I where

$I + I = mr^2$ (by perpendicular axes theorem)

$$\therefore\ I = \frac{mr^2}{2}$$



The moment of inertia of the ring about a tangent is I' where

$I' = I + mr^2$ by the parallel axes theorem

$$\text{i.e. } I' = \frac{mr^2}{2} + mr^2 = \frac{3mr^2}{2}$$

So, if the radius of gyration is k

$$mk^2 = 3\frac{mr^2}{2}$$

$$\text{i.e. } k = \sqrt{\frac{3}{2}}r$$

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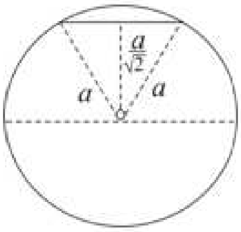
Exercise C, Question 8

Question:

Find the moment of inertia of a uniform solid sphere of radius a and mass m about a chord of the sphere which lies at a distance $\frac{a}{\sqrt{2}}$ from the centre of the sphere.

Solution:



The moment of inertia of the sphere about a diameter $= \frac{2}{5}ma^2$

This is true for any diameter and in particular for a diameter parallel to the chord.

Let I be the moment of inertia of the sphere about the chord, which is a distance $\frac{a}{\sqrt{2}}$ from O

Then

$$I = \frac{2}{5}ma^2 + m\left(\frac{a}{\sqrt{2}}\right)^2 \text{ [parallel axes theorem]}$$
$$= \frac{2}{5}ma^2 + \frac{ma^2}{2}$$
$$= \frac{9}{10}ma^2$$

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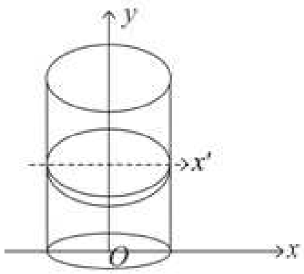
Exercise C, Question 9

Question:

Use calculus to find the moment of inertia of a thin hollow uniform right circular cylinder of mass M, radius R and height H about a diameter of an end circle. The cylinder is open at both ends.

Solution:



Let O be the centre of the circular base of the cylinder and let the x-axis be in the direction of the diameter of the base.
Let the y-axis be the axis of the cylinder.
Divide the cylinder into rings – one of which is shown. Let this ring have radius R, thickness δy and be at a distance y from the x-axis. Its mass is δm.
The moment of inertia of the ring about the y-axis is $\delta m R^2 = 2\pi\rho R\delta y \cdot R^2 = 2\pi\rho R^3\delta y$
Let the moment of inertia of the ring about a diameter perpendicular to the y-axis be $\delta I_{x'}$.
Then $\delta I_{x'} + \delta I_{x'} = 2\pi\rho R^3\delta y$ – using perpendicular axes theorem.
i.e. $\delta I_{x'} = \pi\rho R^3\delta y$
So the moment of inertia of the ring about the x-axis is $\delta I_{x'} + \delta m y^2$, using the parallel axes theorem i.e. $\pi\rho R^3\delta y + 2\pi\rho R y^2\delta y$
Adding all such rings and letting $\delta y \to 0$

$$I_x = \int_0^H \pi\rho R\left(R^2 + 2y^2\right)dy$$
$$= \pi\rho R\left[R^2 y + \frac{2}{3}y^3\right]_0^H$$
$$= \pi\rho R\left[R^2 H + \frac{2}{3}H^3\right]$$

But the cylinder has mass M. So $2\pi RH\rho = M$

$$\therefore\ I_x = \frac{M}{2H}\left[R^2 H + \frac{2}{3}H^3\right]$$
$$= \frac{M}{6}\left[3R^2 + 2H^2\right]$$

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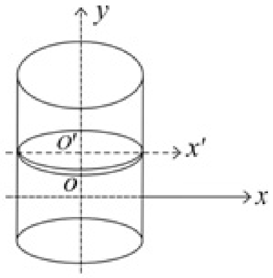
Exercise C, Question 10

Question:

Find the moment of inertia of a solid uniform right circular cylinder of mass M, radius R and height H about an axis through the centre of gravity perpendicular to the axis of the cylinder.

Solution:



Take the y-axis as the axis of the cylinder and the x-axis passes through the centre of gravity as shown.
Divide the cylinder into discs. The disc shown has radius R, thickness δy and is at height y above the x-axis.

The mass per unit volume of the cylinder $= \dfrac{M}{\pi R^2 H}$

$\therefore$ mass of disc $= \dfrac{M}{\pi R^2 H} \cdot \pi R^2 \delta y = \dfrac{M \delta y}{H}$

For the disc $I_y = \left(\dfrac{M\delta y}{H}\right) \times \dfrac{R^2}{2}$

$\therefore$ The M.I. of disc about its diameter $O'x'$, parallel to the x-axis is

$I_{x'} = \left(\dfrac{M\delta y}{H}\right) \times \dfrac{R^2}{4}$ (from the perpendicular axis theorem)

$\therefore$ M.I. of disc about Ox is I_x where

$I_x = I_{x'} + \left(\dfrac{M\delta y}{H}\right) \times y^2$ (from the parallel axes theorem)

$= \dfrac{M\delta y}{H}\left[\dfrac{R^2}{4} + y^2\right]$

The moment of inertia for the cylinder is obtained by adding the moments of inertia for all such discs and letting $\delta y \to 0$

$$\therefore\ I = \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{M}{H}\left(\frac{R^2}{4} + y^2\right) dy$$

$$= \frac{M}{H}\left[\frac{R^2}{4}y + \frac{y^3}{3}\right]_{-\frac{H}{2}}^{\frac{H}{2}}$$

$$= \frac{2M}{H}\left[\frac{R^2 H}{8} + \frac{H^3}{24}\right]$$

$$= \frac{MR^2}{4} + \frac{MH^2}{12}$$

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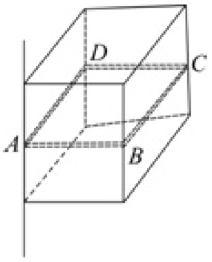
Exercise C, Question 11

Question:

Find the moment of inertia of a uniform cube of mass M and edge a about an axis along one edge.

Solution:



Consider a square cross section $ABCD$ of the cube.
Let its mass be δm.

Its M.I. about $AD = \frac{4}{3}\delta m\left(\frac{a}{2}\right)^2 = \frac{1}{3}\delta ma^2$

Also its M.I. about $AB = \frac{1}{3}\delta ma^2$

$\therefore$ By perpendicular axis theorem, its moment of inertia about an axis through A perpendicular to $ABCD$ is δI where

$\delta I = \frac{1}{3}\delta ma^2 + \frac{1}{3}\delta ma^2 = \frac{2}{3}\delta ma^2$

The M.I. of the cube about the edge through A is obtained by adding all such square cross sections

$$\therefore I = \sum \frac{2}{3}\delta ma^2$$
$$= \frac{2}{3}a^2 \sum \delta m$$
$$= \frac{2}{3}Ma^2$$

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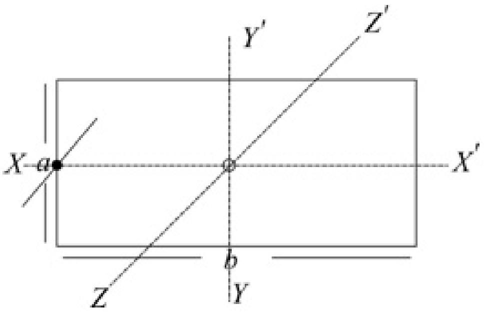
Exercise C, Question 12

Question:

Find the moment of inertia of a uniform rectangular lamina of mass M and sides a and b about an axis, perpendicular to the lamina, through the mid-point of a side of length a.

Solution:



Take XX', YY' and ZZ' as three axes meeting at O, the centre of the rectangle. XX' and YY' are parallel to sides of the rectangle and ZZ' is perpendicular to the rectangle.

Let L be the axis about which you need to find the moment of inertia. L is parallel to ZZ'

$I_{ZZ'} = I_{XX'} + I_{YY'}$ (perpendicular axis theorem)

$$= \frac{1}{3}m\left(\frac{a}{2}\right)^2 + \frac{1}{3}m\left(\frac{b}{2}\right)^2$$

$$= \frac{1}{12}m\left(a^2 + b^2\right)$$

Then $I_L = I_{ZZ'} + m\left(\frac{b}{2}\right)^2$ (parallel axes theorem)

$$= \frac{1}{12}m\left(a^2 + b^2\right) + \frac{mb^2}{4}$$

$$= \frac{1}{12}ma^2 + \frac{1}{3}mb^2$$

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Exercise C, Question 13

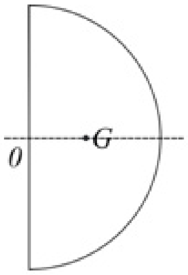
Question:

A uniform semi circular lamina has mass m and radius r.

a State the position of its centre of mass.

b Find the moment of inertia of the lamina about an axis through its centre of mass, perpendicular to the lamina.

Solution:



a On its axis of symmetry at a distance $\frac{4r}{3\pi}$ from O, the mid-point of its straight edge.

b Let the moment of inertia of the semi-circular lamina about an axis perpendicular to the lamina through O be I_O.

Then, as two such laminas make a disc of mass $2m$

$I_O + I_O = \frac{2mr^2}{2}$ – by the additive rule.

$\therefore\ I_O = \frac{mr^2}{2}$

The required moment of inertia I_G' may be obtained by using the parallel axes theorem.

As $I_O = I_G' + m\left(\frac{4r}{3\pi}\right)^2$

$I_G' = \frac{mr^2}{2} - \frac{16mr^2}{9\pi^2}$

$= \frac{mr^2}{18\pi^2}(9\pi^2 - 32)$

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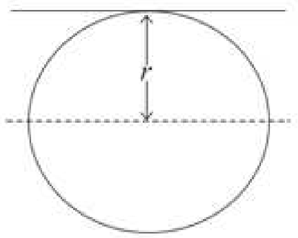
Exercise C, Question 14

Question:

Find the moment of inertia of a uniform solid sphere of mass m and radius r about a tangent at any point on the surface.

Solution:



Moment of inertia about a diameter $= \frac{2}{5}mr^2$

Then using parallel axes theorem:

$$\text{Moment of inertia about tangent} = \frac{2}{5}mr^2 + mr^2$$
$$= \frac{7}{5}mr^2$$

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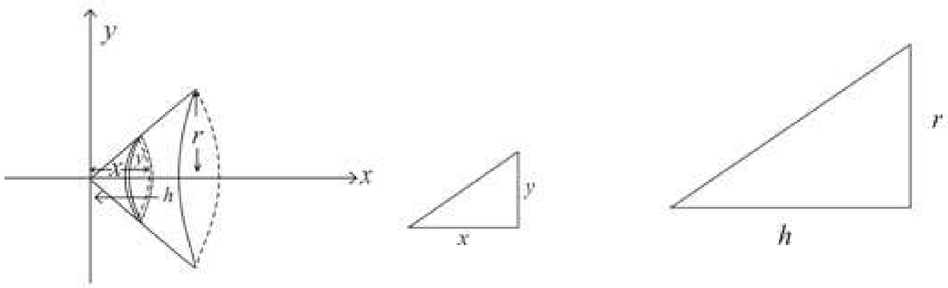
Exercise C, Question 15

Question:

Find, by integration, the moment of inertia of a uniform solid cone of mass m, base radius r and height h about a diameter of the base.

Solution:



Divide the cone into thin discs – one of which is shown.
Its mass is δm, its thickness is δx, its radius is y and it is at a distance x from the y-axis.

The moment of inertia of the disc about the x-axis is $\frac{\delta m y^2}{2}$.

Its M.I. about its diameter $= \frac{\delta m y^2}{4}$ (perpendicular axes theorem)

Its M.I. about a diameter of the base $= \frac{\delta m y^2}{4} + \delta m (h-x)^2$ (parallel axes theorem) *

The mass per unit volume of the cone $= \frac{m}{\frac{1}{3}\pi r^2 h} = \frac{3m}{\pi r^2 h}$

$$\therefore \text{ The mass } \delta m = \frac{3m}{\pi r^2 h} \times \pi y^2 \delta x$$
$$= \frac{3my^2}{r^2 h}\delta x \qquad ①$$

Also by similar triangles: $\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{r}{h}x$ ②

Substituting ① and ② into *,

$$\delta I = \frac{3m}{r^2 h}\left(\frac{y^4}{4} + y^2(h-x)^2\right)\delta x$$

i.e. $\delta I = \frac{3m}{r^2 h}\left(\frac{r^4 x^4}{4h^4} + \frac{r^2 x^2}{h^2}(h-x)^2\right)\delta x$

Let $\delta x \to 0$ and find the total moment of inertia of the cone by integration.

$$\text{So } I = \frac{3m}{r^2 h}\int_0^h \frac{r^4}{4h^4}x^4 + \frac{r^2}{h^2}\left(h^2x^2 - 2hx^3 + x^4\right)dx$$
$$= \frac{3m}{r^2 h}\left[\frac{r^4 x^5}{20h^4} + \frac{r^2 x^3}{3} - \frac{2r^2 x^4}{4h} + \frac{r^2}{h^2}\frac{x^5}{5}\right]_0^h$$
$$= \frac{3m}{r^2 h}\left[\frac{r^4 h}{20} + \frac{r^2 h^3}{3} - \frac{r^2 h^3}{2} + \frac{r^2 h^3}{5}\right]$$
$$= \frac{3mr^2}{20} + \frac{mh^2}{10}$$

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Exercise D, Question 1

Question:

You may assume that the moment of inertia of a uniform circular disc, of mass m *and radius* a, *about an axis through its centre and perpendicular to its plane is* $\frac{1}{2}ma^2$.

A cartwheel is modelled as a uniform circular disc, of mass m and radius a, to which is attached a thin metal circular rim, also of mass m and radius a. The cartwheel rotates about the axis through its centre and perpendicular to its plane.

Find the radius of gyration of the cartwheel about this axis. ***E***

Solution:

Moment of inertia of circular disc $= \frac{1}{2}ma^2$

Moment of inertia of circular rim $= ma^2$

$\therefore$ M.I. of cartwheel $= \frac{1}{2}ma^2 + ma^2$ (additive rule)

$= \frac{3}{2}ma^2$

But mass of cartwheel $= 2m$

$\therefore$ If its radius of gyration $= k$

$$\text{M.I.} = 2mk^2 = \frac{3}{2}ma^2$$

$$\therefore k^2 = \frac{3}{4}a^2$$

$$\text{i.e. } k = \frac{\sqrt{3}a}{2}$$

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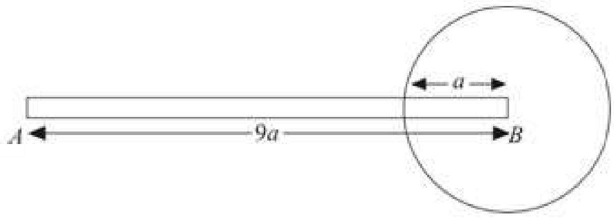
Exercise D, Question 2

Question:



A pendulum P is modelled as a uniform rod AB, of length $9a$ and mass M, rigidly fixed to a uniform circular disc of radius a and mass $2M$. The end B of the rod is attached to the centre of the disc and the rod lies in the plane of the disc as shown in the figure. The pendulum is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through end A and is perpendicular to the plane of the disc.

Show that the moment of inertia of P about L is $190Ma^2$. ***E*** (adapted)

Solution:

Moment of inertia of rod about $L = \frac{4}{3}M\left(\frac{9a}{2}\right)^2$

$= \frac{1}{3}M \times (9a)^2$

$= 27Ma^2$

Moment of inertia of disc about $L = 2M\left(\frac{a^2}{2}\right) + 2M(9a)^2$ (By parallel axes theorem)

$= Ma^2 + 162Ma^2$

$= 163Ma^2$

$\therefore$ Moment of inertia of pendulum about $L = 27Ma^2 + 163Ma^2$ (additive law)

$= 190Ma^2$

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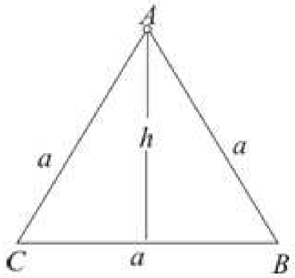
Exercise D, Question 3

Question:

A uniform wire of length $3a$ and mass $3m$ is bent into the shape of an equilateral triangle. Find the moment of inertia of the triangle about an axis through a vertex perpendicular to the plane of the lamina. ***E***

Solution:



By Pythagoras' Theorem:

$$h^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} *$$

The diagram shows the equilateral triangle ABC. Let L be the axis through A, perpendicular to the plane of ABC.

Moment of inertia of AB about $L = \frac{4}{3}m\left(\frac{a}{2}\right)^2$

Moment of inertia AC about $L = \frac{4}{3}m\left(\frac{a}{2}\right)^2$

Moment of inertia CB about $L = \frac{m}{3}\left(\frac{a}{2}\right)^2 + mh^2$ – from parallel axes theorem

∴ By additive rule:
moment of inertia of the triangle about L

$$= \frac{4}{3}m\left(\frac{a}{2}\right)^2 + \frac{4}{3}m\left(\frac{a}{2}\right)^2 + \frac{m}{3}\left(\frac{a}{2}\right)^2 + mh^2$$

$$= \frac{1}{3}ma^2 + \frac{1}{3}ma^2 + \frac{1}{12}ma^2 + \frac{3ma^2}{4} \text{ (from*)}$$

$$= \frac{18}{12}ma^2$$

$$= \frac{3}{2}ma^2$$

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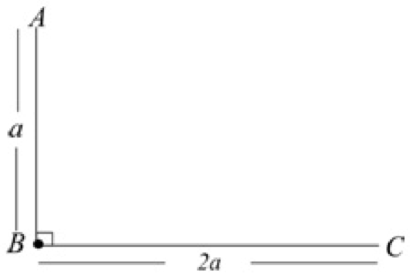
Exercise D, Question 4

Question:

A uniform piece of wire ABC, of total length $3a$ and mass m, is bent to form a right angle at B, with straight arms AB and BC of length a and $2a$ respectively. Show that the moment of inertia of the wire about the axis L through B perpendicular to the plane of the wire is ma^2. ***E***

Solution:



As the wire is uniform, mass of AB is $\frac{m}{3}$ and mass of BC is $\frac{2m}{3}$.

Moment of inertia of AB about $L = \frac{4}{3}\left(\frac{m}{3}\right)\left(\frac{a}{2}\right)^2$

Moment of inertia BC about $L = \frac{4}{3}\left(\frac{2m}{3}\right)\left(\frac{2a}{2}\right)^2$

$\therefore$ Moment of inertia of wire about $L = \frac{4}{3} \times \frac{ma^2}{12} + \frac{4}{3} \times \frac{8ma^2}{12}$

$= \frac{ma^2}{9} + \frac{8ma^2}{9}$

$= ma^2$

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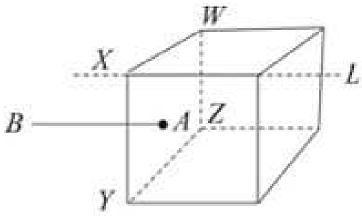
Exercise D, Question 5

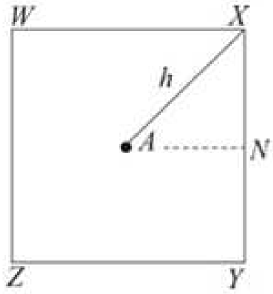
Question:

A thin uniform rod of mass m and length $2l$ is attached at one end to the centre of a face of a uniform solid cube of mass $8m$ and side l. The rod is perpendicular to the face to which it is attached. Find the moment of inertia of the system about an edge of the cube which is parallel to the rod. ***E***

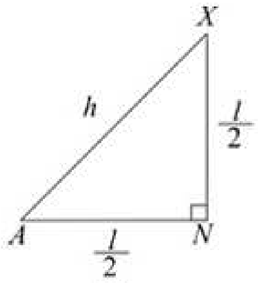
Solution:



i



ii



iii

Diagram **i** shows the rod AB attached at A, the centre of the face $WXYZ$. It also shows the axis L through point X perpendicular to face $WXYZ$.
Diagram **ii** shows the face $WXYZ$ and diagram **iii** shows an enlargement of ΔANX, where N is the mid-point of the edge XY

Let $AX = h$ where $h^2 = \left(\frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2 = \frac{2l^2}{4} = \frac{l^2}{2}$ (from Pythagoras' Theorem)

Let mass of square be m'
moment of inertia of the rod AB about axis L
$= mh^2$
$= \frac{ml^2}{2}$ ①

moment of inertia of cube about $L =$ moment of inertia of square $WXYZ$ of same mass about L (stretching rule)

Moment of inertia of square about axis along $AN = \frac{1}{3}m'\left(\frac{l}{2}\right)^2$

Also M.I. of square about axis perpendicular to AN in plane of square $= \frac{1}{3}m'\left(\frac{l}{2}\right)^2$

$\therefore$ By perpendicular axes theorem M.I. of square about axis through A perpendicular to plane $= \frac{1}{3}m'\frac{l^2}{4} + \frac{1}{3}m'\frac{l^2}{4} = \frac{1}{6}m'l^2$

By parallel axes theorem M.I. of $WXYZ$ about L
$= \frac{1}{6}m'l^2 + m'h^2$
$= \frac{1}{6}m'l^2 + \frac{1}{2}m'l^2$
$= \frac{2}{3}m'l^2$

So M.I. of cube about $L = \frac{2}{3} \times 8ml^2$ ②

Using results ① and ② the M.I. of the system about L
$= \frac{ml^2}{2} + \frac{16}{3}ml^2$
$= \frac{35}{6}ml^2$

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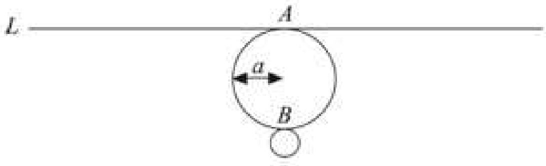
Exercise D, Question 6

Question:



A uniform disc has mass m and radius a.

a Show that the moment of inertia of the disc about a tangent L lying in the plane of the disc is $\frac{5}{4}ma^2$.

The line L is a tangent to the disc at the point A, and AB is a diameter of the disc, as shown in the figure. A particle of mass m is attached to the disc at B.

b Find the moment of inertia of the loaded disc about the tangent L. ***E***

Solution:

a M.I. of disc about axis through its centre, perpendicular to its plane $= \frac{ma^2}{2}$ ← This is a result you may quote.

$\therefore$ M.I. of disc about diameter $= \frac{ma^2}{4}$ ← From perpendicular axis theorem – as $I + I = \frac{ma^2}{2}$, then $I = \frac{ma^2}{4}$.

$\therefore$ M.I. of disc about tangent $= \frac{ma^2}{4} + ma^2$ (Parallel axes theorem)

i.e. M.I. $= \frac{5ma^2}{4}$

b $I = \frac{5ma^2}{4} + m(2a)^2$

$= \frac{21ma^2}{4}$ (Additive rule)

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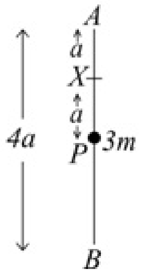
Exercise D, Question 7

Question:

A uniform rod AB of mass m and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis l through the point X on the rod, where $AX = a$. The rod is hanging at rest with B below A when it is struck at its mid-point by a particle P of mass $3m$ moving horizontally with speed u in a direction perpendicular to l. Immediately after the impact P adheres to the rod. Show that after the impact, the moment of inertia about l of the rod and the particle together is $\frac{16}{3}ma^2$. ***E***

Solution:



Moment of inertia of rod about axis perpendicular to it, through mid-point $= \frac{m(2a)^2}{3} = \frac{4ma^2}{3}$

$\therefore$ M.I. of rod about axis l, through $X = \frac{4ma^2}{3} + ma^2$ (parallel axes theorem)

Moment of inertia of particle P about $l = 3ma^2$

$\therefore$ M.I. of rod and particle together $= \frac{4ma^2}{3} + ma^2 + 3ma^2$

$= \frac{16ma^2}{3}$

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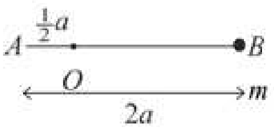
Exercise D, Question 8

Question:

A uniform rod AB has mass m and length $2a$. A particle of mass m is attached to the end B. The loaded rod is free to rotate about a fixed smooth horizontal axis L, perpendicular to the rod and passing through a point O of the rod, where $AO = \frac{1}{2}a$.

Show that the moment of inertia of the loaded rod about L is $\frac{17ma^2}{6}$. **E**

Solution:



Moment of inertia of rod about mid-point $= \frac{1}{3}ma^2$

$\therefore$ M.I. of rod about axis L, through $O = \frac{1}{3}ma^2 + m\left(\frac{1}{2}a\right)^2$ (parallel axes theorem)

M.I. of particle at B about $L = m\left(\frac{3}{2}a\right)^2$

$$\therefore \text{ M.I. of the loaded rod} = \frac{1}{3}ma^2 + \frac{1}{4}ma^2 + \frac{9}{4}ma^2$$
$$= \frac{34}{12}ma^2$$
$$= \frac{17ma^2}{6}$$

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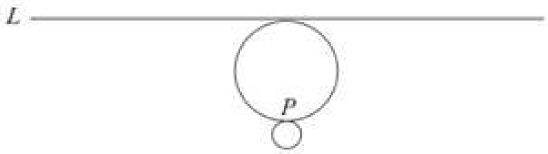
Exercise D, Question 9

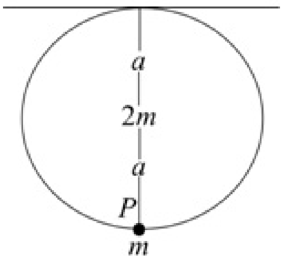
Question:



An ear-ring is modelled as a uniform solid sphere of mass $2m$ and radius a, with a particle of mass m attached to a point P on the surface of the sphere. The ear- ring is free to rotate about a fixed horizontal axis L which is tangential to the sphere and passes through a point diametrically opposite to P, as shown in the figure.

Show that the moment of inertia of the ear- ring about L is $\frac{34}{5}ma^2$. ***E***

Solution:



Moment of inertia of sphere about diameter $= \frac{2}{5}(2m)a^2$

$= \frac{4ma^2}{5}$

$\therefore$ M.I. of sphere about $L = \frac{4}{5}ma^2 + 2ma^2$ (parallel axes theorem)

$= \frac{14}{5}ma^2$

M.I. of particle at P about $L = m(2a)^2$

$= 4ma^2$

$\therefore$ Total M.I. of ear-ring about $L = \frac{14}{5}ma^2 + 4ma^2$ (additive rule)

i.e. M.I. $= \frac{34}{5}ma^2$

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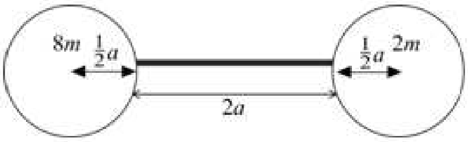
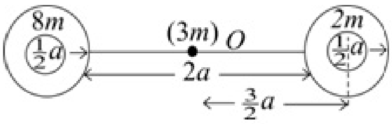
Exercise D, Question 10

Question:



A model of a timing device in a clock consists of a uniform rod, of mass $3m$ and length $2a$, the ends of which are attached to two uniform solid spheres, each of radius $\frac{1}{2}a$ as shown in the figure. One sphere has mass $8m$ and the other has mass $2m$. The device rotates freely in a vertical plane about a horizontal axis through the centre of the rod and perpendicular to it. Show that the moment of inertia of the system about this axis is $\frac{49}{2}ma^2$. ***E***

Solution:



Let O be the centre of the rod and let L be the horizontal axis through O, perpendicular to the rod.

M.I. of rod about $L = \frac{1}{3}(3m)a^2 = ma^2$

M.I. of sphere mass $2m$ about its diameter $= \frac{2}{5} \times 2m\left(\frac{1}{2}a\right)^2$

$= \frac{1}{5}ma^2$

$\therefore$ M.I. of sphere mass $2m$ about $L = \frac{1}{5}ma^2 + 2m\left(\frac{3}{2}a\right)^2$ (parallel axes theorem)

$= \frac{47}{10}ma^2$

Similarly M.I. of sphere mass $8m$ about $L = \frac{2}{5} \times 8m\left(\frac{1}{2}a\right)^2 + 8m\left(\frac{3}{2}a\right)^2$

$= \frac{188}{10}ma^2$

Using the additive law:

$\therefore$ M.I. of whole timing device $= ma^2 + \frac{47}{10}ma^2 + \frac{188}{10}ma^2$

$= \frac{49ma^2}{2}$

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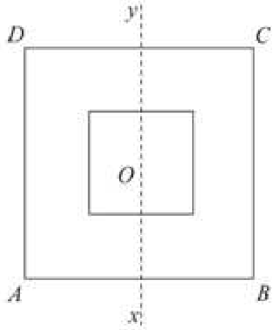
Exercise D, Question 11

Question:



A uniform lamina of mass m is formed from a square lamina $ABCD$ of side $2a$ by cutting out a square of side a. Both squares have the same centre O and their sides are parallel as shown in the figure. The points X and Y are the mid-points of AB and CD respectively.

a Find the moment of inertia of the lamina about an axis passing through X and Y.

b Hence find the radius of gyration of the lamina about an axis perpendicular to its plane passing through O. ***E***

Solution:

a Let mass per unit area be ρ.

Moment of inertia of $ABCD$ about $XY = \frac{1}{3}(\rho \times 4a^2) \times a^2$

Moment of inertia of smaller square about $XY = \frac{1}{3}(\rho \times a^2) \times \left(\frac{a}{2}\right)^2$

$\therefore$ By additive law moment of inertia of lamina $= \frac{1}{3}\rho \times 4a^4 - \frac{1}{3}\rho \times \frac{a^4}{4}$

$= \frac{1}{3}\rho a^4\left(4 - \frac{1}{4}\right)$

$= \frac{5}{4}\rho a^4$

But $m = \rho(4a^2 - a^2)$

i.e. $\rho = \frac{m}{3a^2}$

$\therefore$ M.I. $= \frac{5ma^2}{12}$

b Moment of inertia about axis perpendicular to plane

$= \frac{5ma^2}{12} + \frac{5ma^2}{12}$ (by perpendicular axes theorem)

$= \frac{5ma^2}{6}$

But M.I. $= mk^2$ where k is the radius of gyration – so $k^2 = \frac{5a^2}{6}$ and $k = \sqrt{\frac{5}{6}}a$

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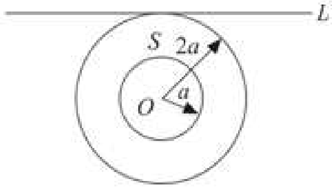
Exercise D, Question 12

Question:



A lamina S is formed from a uniform disc, centre O and radius $2a$, by removing the disc of centre O and radius a, as shown. The mass of S is M.

a Show that the moment of inertia of S about an axis through O and perpendicular to its plane is $\frac{5}{2}Ma^2$.

The lamina is free to rotate about a fixed smooth horizontal axis L. The axis L lies in the plane of S and is a tangent to its outer circumference, as shown.

b Show that the moment of inertia of S about L is $\frac{21}{4}Ma^2$. ***E*** (adapted)

Solution:

a Let the mass per unit area be ρ and let the axis through O perpendicular to the lamina be L.

The moment of inertia of the disc with radius $2a$ about L

$$= \left[\rho \cdot \pi(2a)^2\right] \times \frac{(2a)^2}{2} = 8\pi\rho a^4$$

The moment of inertia of the disc with radius a about $L = \rho\pi a^2 \times \frac{a^2}{2} = \frac{1}{2}\rho\pi a^4$

$\therefore$ Moment of inertia of lamina $S = 8\pi\rho a^4 - \frac{1}{2}\pi\rho a^4$

$$= \frac{15}{2}\pi\rho a^4 \quad ①$$

But the mass of $S = M = \pi\rho[(2a)^2 - a^2]$

$$= 3\pi\rho a^2 \quad ②$$

Substitute $\rho = \frac{M}{3\pi a^2}$ into equation ① to give M.I. $= \frac{5}{2}Ma^2$

b Let the moment of inertia of S about a diameter parallel to L be I.

The moment of inertia of S about a diameter perpendicular to L is also I.

Then $I + I = \frac{5}{2}Ma^2$ (from perpendicular axes theorem)

$\therefore I = \frac{5}{4}Ma^2$

The moment of inertia of S about $L = \frac{5}{4}Ma^2 + M(2a)^2$ (from the parallel axes theorem)

$\therefore$ Required moment of inertia $= \frac{5}{4}Ma^2 + 4Ma^2$

$$= \frac{21}{4}Ma^2$$

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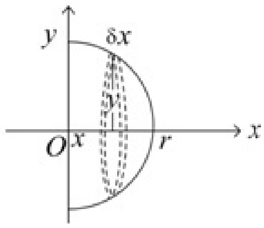
Exercise D, Question 13

Question:

Use integration to show that the radius of gyration of a uniform solid hemisphere of mass m and radius r about a diameter of the circular base is $\sqrt{\frac{2}{5}}r$. ***E***

Solution:



Let the mass per unit volume be ρ.
Divide the hemisphere up into discs of radius y, thickness δx at a distance x from the circular base.

The M.I. of the disc shown about $Ox = (\rho\pi y^2 \delta x) \times \dfrac{y^2}{2}$

$\therefore$ The M.I. of the disc about its diameter parallel to $Oy = (\rho\pi y^2 \delta x)\dfrac{y^2}{4}$

(perpendicular axes theorem)

$\therefore$ Its M.I. about $Oy = \rho\pi y^2 \delta x \dfrac{y^2}{4} + \rho\pi y^2 \delta x \cdot x^2$ (parallel axes theorem)

Summing all such discs and letting $\delta x \to 0$ gives I, the moment of inertia of the hemisphere.

So $I = \rho\pi \int_0^r \dfrac{y^4}{4} dx + \rho\pi \int_0^r y^2 x^2 \, dx$

But $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$$\therefore I = \rho\pi \int_0^r \frac{1}{4}(r^4 - 2r^2x^2 + x^4) + (r^2x^2 - x^4) dx$$

$$= \rho\pi \left[\frac{1}{4}\left(r^4 x - \frac{2}{3}r^2 x^3 + \frac{1}{5}x^5 \right) + \frac{1}{3}r^2x^3 - \frac{1}{5}x^5 \right]_0^r$$

$$= \rho\pi \left[\frac{1}{4}r^5 - \frac{1}{6}r^5 + \frac{1}{20}r^5 + \frac{1}{3}r^5 - \frac{1}{5}r^5 \right]$$

$$= \frac{\rho\pi}{60}[15 - 10 + 3 + 20 - 12]r^5$$

$$= \frac{4\rho\pi r^5}{15}$$

But the mass of the hemisphere $m = \dfrac{2}{3}\pi\rho r^3 \Rightarrow \rho = \dfrac{3m}{2\pi r^3}$

$\therefore I = \dfrac{2}{5}mr^2$

and the radius of gyration $k = \sqrt{\dfrac{2}{5}}r$

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Exercise D, Question 14

Question:

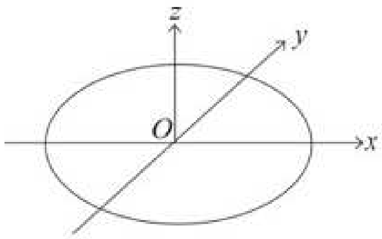
Assuming that the moment of inertia of a uniform circular disc, of mass m and radius r, about an axis through its centre and perpendicular to its plane is $\frac{1}{2}mr^2$,

a deduce that its moment of inertia about a diameter is $\frac{1}{4}mr^2$.

b Hence, using integration, show that the moment of inertia of a uniform solid circular cylinder, of mass M, radius r and height h, about a diameter of one of its plane faces is $\frac{1}{12}M(3r^2+4h^2)$. **E**

Solution:

a



Let O be the centre of the circular disc. Take axes Ox and Oy in the plane of the disc and Oz perpendicular to the disc.

Then $I_{Oz} = \frac{1}{2}mr^2$

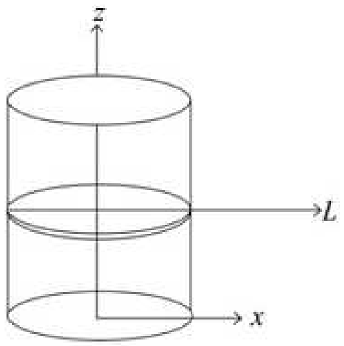
Also $I_{Ox} + I_{Oy} = I_{Oz}$ (perpendicular axes theorem)

Let $I_{Ox} = I$, then $I_{Oy} = I$ also (symmetry)

$\therefore 2I = \frac{1}{2}mr^2$

So $I = \frac{1}{4}mr^2$

b



Consider the cylinder divided up into a large number of thin discs.
Let a typical disc have radius r, thickness δz and be at a distance z from the Oxy plane.

The M.I. of this disc is $\frac{mr^2}{4}$ about its diameter in the direction L, parallel to Ox, where m is the mass of the disc.

Its M.I. about a diameter of the base of the cylinder, Ox is $\frac{mr^2}{4}+mz^2$
(by parallel axes theorem)

As the cylinder is uniform $\frac{m}{M}=\frac{\delta z}{h}$

$$\therefore m=\frac{M}{h}\delta z$$

So M.I. of cylinder about base diameter is obtained from $\sum \frac{M}{h}\left[\frac{r^2}{4}+z^2\right]\delta z$ as $\delta z \to 0$

i.e.

$$I=\frac{M}{h}\int_0^h \frac{r^2}{4}+z^2 \, \mathrm{d}z$$

$$=\frac{M}{h}\left[\frac{r^2}{4}z+\frac{1}{3}z^3\right]_0^h=\frac{M}{h}\left[\frac{r^2h}{4}+\frac{h^3}{3}\right]$$

$$=\frac{M}{12}\left[3r^2+4h^2\right]$$

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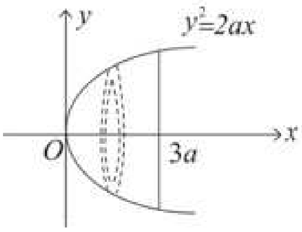
Exercise D, Question 15

Question:

You may assume, without proof, that the moment of inertia of a uniform circular disc, of mass m *and radius* r, *about an axis through its centre and perpendicular to its plane is* $\frac{1}{2}\text{mr}^2$.

A uniform solid S is generated by rotating the finite region bounded by the curve with equation $y^2 = 2ax$ and the line with equation $x = 3a$ through $180°$ about the x-axis. The volume of S is $9\pi a^3$ and its mass is M. Show, by integration, that the moment of inertia of S about its axis of symmetry is $2Ma^2$. ***E***

Solution:



Let the mass per unit volume be ρ. Divide the solid S into a large number of thin discs, perpendicular to the x-axis. A typical disc is shown. This has mass $\rho\pi y^2\delta x$ and its moment of inertia about the x-axis is $\rho\pi y^2\delta x \times \frac{y^2}{2}$

$\therefore$ By summation and letting $\delta x \to 0$ the moment of inertia of S about the axis of symmetry

$$I = \frac{\rho\pi}{2}\int_0^{3a} y^4 \, dx$$

$$= \frac{\rho\pi}{2}\int_0^{3a} (2ax)^2 \, dx$$

$$= \frac{4a^2\rho\pi}{2}\left[\frac{x^3}{3}\right]_0^{3a}$$

$$= \frac{4a^5\rho\pi \times 9}{2}$$

$$= 18a^5\rho\pi \quad ①$$

But as the volume of S is $9\pi a^3$

$\therefore M = \rho \cdot 9\pi a^3$

$\therefore I = 2Ma^2$ by substitution into equation ①

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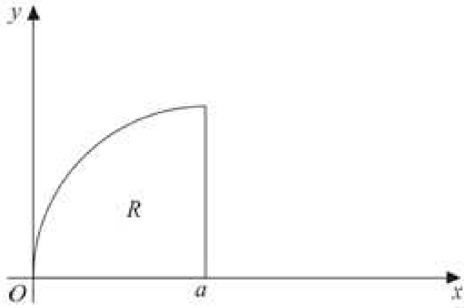
Exercise D, Question 16

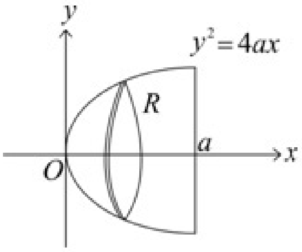
Question:

You may assume, without proof, that the moment of inertia of a uniform disc, of mass m *and radius* r*, about an axis through its centre perpendicular to its plane is* $\frac{1}{2}\text{mr}^2$.



A region R is bounded by the curve $y^2 = 4ax (y > 0)$, the x-axis and the line $x = a$ $(a > 0)$, as shown. A uniform solid S of mass M is formed by rotating R about the x-axis through $360°$. Using integration, prove that the moment of inertia of S about the x-axis is $\frac{4}{3}Ma^2$. **E**

Solution:



Divide S into discs parallel to the circular base of the solid let the mass per unit volume be ρ.

Then $M = \rho \int_0^a \pi y^2 \, dx$

$= \rho \int_0^a \pi \cdot 4ax \, dx$

$= \left[2\pi\rho a x^2\right]_0^a$

$= 2\pi\rho a^3$

$\therefore \rho = \dfrac{M}{2\pi a^3}$ *

Also $I = \rho \int_0^a \pi y^2 \times \dfrac{y^2}{2} \, dx$

$= \dfrac{\pi\rho}{2} \int_0^a (4ax)^2 \, dx$

$= 8a^2\pi\rho \left[\dfrac{x^3}{3}\right]_0^a$

$= \dfrac{8}{3} a^5 \pi\rho$

Substitute the value of ρ from *

Then $I = \dfrac{4}{3} Ma^2$

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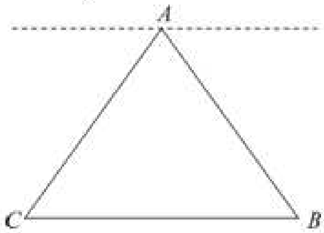
Exercise D, Question 17

Question:

a Show by integration, that the moment of inertia of a uniform rod, of length $2L$ and mass m, about an axis through the centre of the rod and inclined at an angle θ to the rod is $\frac{1}{3}mL^2\sin^2\theta$.



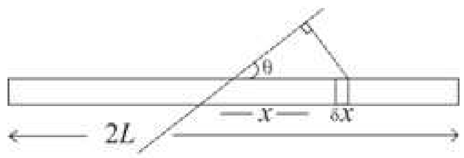
A framework in the shape of an equilateral triangle ABC is formed from three uniform rods, each of length $2L$ and mass m, as shown in the figure.

b Find the moment of inertia of the framework about an axis in the plane of the framework, parallel to BC and passing through A.

c Hence find the radius of gyration of the framework about this axis. ***E***

Solution:

a



Divide the rod into small pieces of length δx at a distance x along the rod from the middle. The perpendicular distance from the small piece shown to the axis is $x\sin\theta$, where θ is the constant given angle.

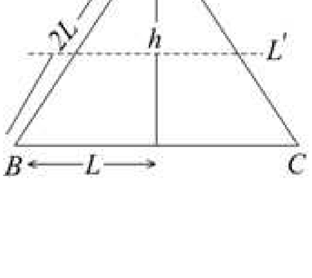
The mass of the small piece is $\frac{m}{2L}\delta x$

You obtain $I = \int_{-L}^{L}\frac{mx^2\sin^2\theta}{2L}dx$

i.e. $I = \frac{m}{2L}\sin^2\theta\left[\frac{x^3}{3}\right]_{-L}^{L}$

$= \frac{mL^2\sin^2\theta}{3}$

b



Moment of inertia of AB about axis L' shown

$$= \frac{mL^2 \sin^2 60^\circ}{3} = \frac{mL^2}{4} \qquad \text{(from result in } \mathbf{a}\text{)}$$

Moment of inertia of AC about axis L' shown $= \frac{mL^2}{4}$ also

By parallel axis theorem, M.I. of AB about axis L = M.I. of AC about axis

$$L = \frac{mL^2}{4} + m\left(\frac{h}{2}\right)^2 *$$

Moment of inertia of BC about given axis $= mh^2$ where $h^2 = (2L)^2 - L^2$

(from Pythagoras' Theorem)

i.e. $h^2 = 3L^2$

So for BC moment of inertia about axis $L = 3mL^2$ and for AB and AC, each moment of inertia about L

$$= \frac{mL^2}{4} + \frac{3mL^2}{4}$$

$$= mL^2 \qquad \text{(from *)}$$

So the moment of inertia of the framework, by the additive rule,

$$= mL^2 + mL^2 + 3mL^2$$

$$= 5mL^2$$

c Let the radius of gyration of the framework be k. As its mass $= 3m$

$\therefore$ its moment of inertia $= 3mk^2 = 5mL^2$

$$\text{i.e. } k^2 = \frac{5}{3}L^2$$

$$\therefore k = \sqrt{\frac{5}{3}}L$$

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Exercise A, Question 1

Question:

A uniform circular disc of mass 2 kg and radius 0.7 m is rotating in a horizontal plane about a smooth fixed vertical axis through its centre. Calculate its kinetic energy when it is rotating at 5 rad s^{-1}.

Solution:

$$\text{K.E.} = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}\times\left(\frac{1}{2}\times 2\times 0.7^2\right)\times 5^2$$

$$= 6.125 \text{ J}$$

The M.I. of a circular disc is in the formula book.

The kinetic energy is 6.125 J.

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Exercise A, Question 2

Question:

A uniform circular disc of mass 4 kg and radius 0.25 m has particles of mass 0.1 kg, 0.2 kg and 0.8 kg attached to it at points which are 0.2 m, 0.1 m and 0.15 m respectively from the centre of the disc. The loaded disc is rotating at 4 rad s^{-1} about a fixed smooth vertical axis through its centre perpendicular to the disc.

a Calculate the kinetic energy of the loaded disc.

The disc is now brought to rest.

b Write down the work done by the retarding force.

Solution:

a M.I. of disc and particles $= \frac{1}{2} \times 4 \times 0.25^2 + 0.1 \times 0.2^2 + 0.2 \times 0.1^2 + 0.8 \times 0.15^2$

$= 0.149 \text{ kg m}^2$

$\text{K.E.} = \frac{1}{2} I\omega^2$

$= \frac{1}{2} \times 0.149 \times 4^2$

$= 1.192 \text{ J}$

The kinetic energy is 1.19 J (3 s.f.)

b The work done by the retarding force is 1.19 J (3 s.f.) ← Work done by the retarding force = loss of K.E.

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Exercise A, Question 3

Question:

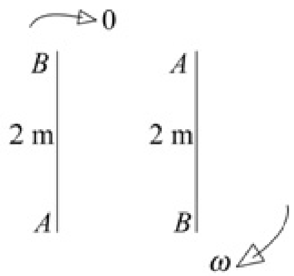
A uniform rod AB of mass 2.5 kg and length 2 m can rotate in a vertical plane about a fixed smooth horizontal axis through A perpendicular to AB. Initially it is at rest with B vertically above A. It is then slightly disturbed and begins to rotate.

a Calculate the potential energy lost by the rod when it is horizontal.

b Write down the kinetic energy of the rod when it is horizontal.

c Calculate the angular speed of the rod when B is vertically below A.

Solution:



a P.E. lost $= mgh = 2.5 \times 9.8 \times 1$

$= 24.5$

The potential energy lost is 24.5 J

b The kinetic energy of the rod when it is horizontal is 24.5 J

c M.I. of the rod about the axis through A

$$= \frac{4}{3} \times 2.5 \times 1^2$$

The formula for the required M.I. can be obtained from the formula book.

$$= \frac{10}{3}$$

$$\frac{1}{2} I\omega^2 = 2.5g \times 2$$

K.E. gained = P.E. lost

$$\frac{1}{2} \times \frac{10}{3} \omega^2 = 2.5g \times 2$$

You can work from the start or from the horizontal position. The former is easier.

$$\omega^2 = \frac{5 \times 9.8 \times 6}{10}$$

$$\omega = 5.422\ldots$$

The angular speed is 5.42 rad s^{-1} (3 s.f.)

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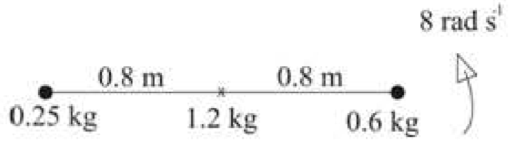
Exercise A, Question 4

Question:

A uniform rod of length 1.6 m and mass 1.2 kg has particles of mass 0.25 kg and 0.6 kg attached, one at each end. The rod is rotating about a fixed smooth vertical axis perpendicular to the rod with angular speed 8 rad s^{-1}. Calculate the kinetic energy of the rod when the axis passes through the mid-point of the rod.

Solution:



M.I. of rod and particles about the given axis through the mid-point

$$= \frac{1}{3} \times 1.2 \times 0.8^2 + 0.25 \times 0.8^2 + 0.6 \times 0.8^2$$

$$= 0.8 \text{ kg m}^2$$

$$\text{K.E.} = \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.8 \times 8^2$$

$$= 25.6$$

The kinetic energy is 25.6 J

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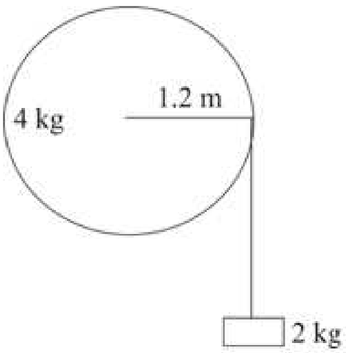
Exercise A, Question 5

Question:

A pulley wheel of mass 4 kg and radius 1.2 m is free to rotate in a vertical plane about a fixed smooth horizontal axis through the centre of the pulley and perpendicular to the pulley. A block of mass 2 kg hangs freely attached to one end of a rope. The other end of the rope is attached to a point on the rim of the pulley and the rope is wound several times around the pulley. Initially the block is hanging 5 m above horizontal ground. The block is then released from rest. The pulley wheel can be modelled as a uniform disc, the block as a particle and the rope as a light inextensible string. Calculate the angular speed of the pulley at the instant when the block hits the ground.

Solution:



P.E. lost by block $= 2g \times 5$
$= 98\text{ J}$

The block falls 5 m.

K.E. gained by block and pulley $= \frac{1}{2}I\omega^2 + \frac{1}{2}m(r\omega)^2$

When the angular speed of the pulley is ω, the block's (linear speed) is $r\omega$.

$$\therefore \frac{1}{2}\times\left(\frac{1}{2}\times 4\times 1.2^2\right)\omega^2 + \frac{1}{2}\times 2\times 1.2^2\omega^2 = 98$$

$$1.44\omega^2 + 1.44\omega^2 = 98$$

$$\omega^2 = \frac{98}{2.88}$$

$$\omega = 5.833\ldots$$

The angular speed is 5.83 rad s^{-1} (3 s.f.)

Solutionbank M5

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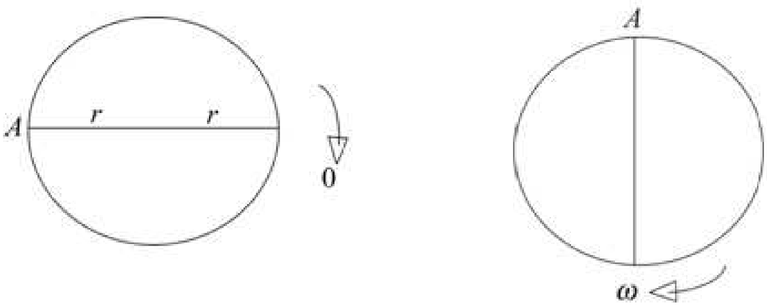
Exercise A, Question 6

Question:

A uniform disc of radius r is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the disc through a point A of its edge. The disc is released from rest with the diameter through A horizontal. Find the angular speed of the disc when this diameter is vertical.

Solution:



$\frac{1}{2}I\omega^2 = mgh$ ← K.E. gained = P.E. lost

M.I. of disc about axis through A

$= \frac{1}{2}mr^2 + mr^2$ ← Use the parallel axes theorem.

$\frac{1}{2} \times \frac{3}{2}mr^2\omega^2 = mgr$ ← Use m for the mass of the disc. It will cancel.

$\omega^2 = \frac{4g}{3r}$

$\omega = 2\sqrt{\frac{g}{3r}}$

The angular speed is $2\sqrt{\frac{g}{3r}}$ ← Any equivalent answer is acceptable.

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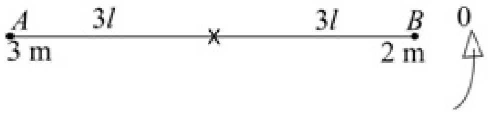
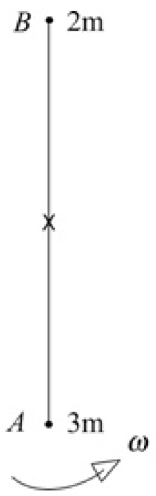
Exercise A, Question 7

Question:

A uniform rod AB of mass m and length $6l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB through its mid-point. Particles of masses $3m$ and $2m$ are attached to ends A and B respectively. The rod is held at rest with AB horizontal and then released. Find, in terms of l and g, the angular speed of the rod when AB is vertical.

Solution:





M.I. of rod and particles about given axis

$$= \frac{1}{3}m(3l)^2 + 3m(3l)^2 + 2m(3l)^2$$

$$= 48\,ml^2$$

$$\frac{1}{2}I\omega^2 = mgh$$ ← K.E. gained = P.E. lost

$$\frac{1}{2} \times 48ml^2\omega^2 = 3mg \times 3l - 2mg \times 3l$$

$$24ml^2\omega^2 = 3mgl$$

$$\omega^2 = \frac{g}{8l}$$

$$\omega = \sqrt{\frac{g}{8l}} = \frac{1}{2}\sqrt{\frac{g}{2l}}$$ ← Any equivalent answer is acceptable.

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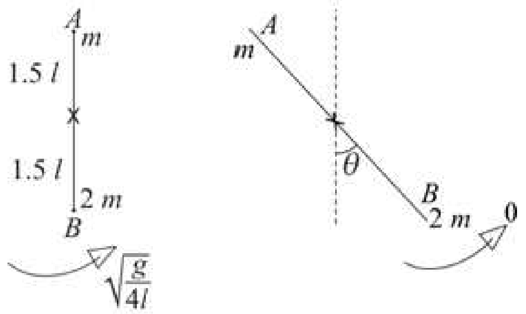
Exercise A, Question 8

Question:

A uniform rod AB of mass m and length $3l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB through its mid-point. Particles of masses m and $2m$ are attached to ends A and B respectively. The rod is initially vertical with B below A.

It then receives an impulse and starts to rotate with angular speed $\sqrt{\frac{g}{4l}}$. Calculate, to the nearest degree, the angle between AB and the downward vertical when the rod first comes to rest.

Solution:



M.I. of loaded rod about given axis

$$= \frac{1}{3}m \times (1.5l)^2 + m \times (1.5l)^2 + 2m \times (1.5l)^2$$

$$= 7.5ml^2$$

Rod comes to rest when the angle between AB and the downward vertical is θ:

$$2mg \times 1.5l(1-\cos\theta) - mg \times 1.5l(1-\cos\theta) = \frac{1}{2} \times 7.5ml^2\left(\sqrt{\frac{g}{4l}}\right)^2$$ ← P.E. gained = K.E. lost

$$1.5\,mlg(1-\cos\theta) = 3.75ml^2 \times \frac{g}{4l}$$

$$1.5(1-\cos\theta) = \frac{3.75}{4}$$

$$1-\cos\theta = \frac{3.75}{6}$$

$$\cos\theta = 1-\frac{3.75}{6}$$

$$\theta = 67.97\ldots$$

The angle is 68° (nearest degree)

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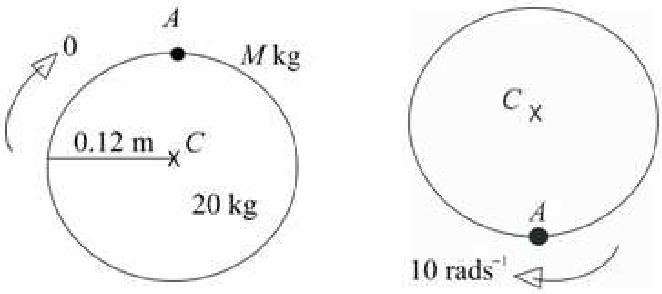
Exercise A, Question 9

Question:

A uniform circular disc of mass 20 kg and radius 12 cm is free to rotate about a fixed smooth horizontal axis through its centre C perpendicular to the disc. A particle of mass M kg is attached to point A of the rim of the disc. Initially the disc is at rest with A vertically above C. The disc is then slightly disturbed. The greatest angular speed of the disc in the subsequent motion is 10 rad s^{-1}. Find the value of M.

Solution:



Watch the units!

M.I. of disc and particle about the given axis

$$= \frac{1}{2} \times 20 \times 0.12^2 + M \times 0.12^2$$
$$= 0.144 + 0.0144M$$

The greatest angular speed will occur when A is vertically below C as the loss of P.E. is greatest here.

$$\frac{1}{2} I\omega^2 = Mg \times 0.24$$

Gain of KE = loss of PE

$$\frac{1}{2}(0.144 + 0.0144M) \times 10^2 = Mg \times 0.24$$
$$14.4 + 1.44M = M \times 9.8 \times 0.48$$
$$M = \frac{14.4}{(9.8 \times 0.48 - 1.44)}$$
$$M = 4.411\ldots$$
$$\therefore M = 4.41 \text{ (3 s.f.)}$$

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Exercise A, Question 10

Question:

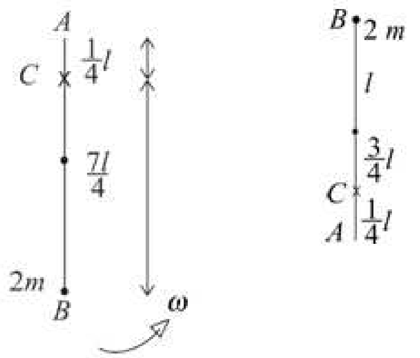
A uniform rod AB is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB through point C of the rod, where $AC = \frac{1}{4}l$. The rod has mass m and length $2l$, and a particle of mass $2m$ is attached to end B. Initially the rod is hanging in equilibrium with B vertically below A. The rod then receives an impulse and starts to rotate with angular speed ω. In the subsequent motion, the rod moves in a complete circle. The least possible value of ω is Ω.

a Show that $\Omega = 4\sqrt{\frac{51g}{337l}}$.

The initial angular speed is 2Ω.

b Find the speed of the particle as it passes vertically above C.

Solution:



a M.I. of rod and particle about given axis through C

$= \frac{1}{3}ml^2 + m \times \left(\frac{3}{4}l\right)^2 + 2m \times \left(\frac{7l}{4}\right)^2$

$= \frac{337}{48}ml^2$

For least ω, angular speed = 0 when B is vertically above A.

At top: P.E. gained $= mg \times 2 \times \frac{3}{4}l + 2mg \times 2 \times \frac{7}{4}l$

$= \frac{17}{2}mgl$

$\therefore \frac{1}{2} \times \frac{337}{48}ml^2\Omega^2 = \frac{17}{2}mgl$

$\Omega^2 = \frac{48}{337} \times 17\frac{g}{l}$

$\Omega = 4\sqrt{\frac{51g}{337l}}$

Use the parallel axes theorem.

K.E. lost = P.E. gained

b $\frac{17mgl}{2} = \frac{1}{2} \times \left(\frac{337}{48}ml^2\right) \times (2\Omega)^2 - \frac{1}{2} \times \left(\frac{337}{48}ml^2\right)\omega^2$

$\frac{337}{48}ml^2\omega^2 = \frac{337}{48}ml^2 \times 4\Omega^2 - 17mgl$

$\frac{337}{48}ml^2\omega^2 = 4 \times 17mgl - 17mgl$

$\frac{337}{48}l\omega^2 = 51g$

$\omega^2 = \frac{48 \times 51g}{337l}$

$\omega = 12\sqrt{\frac{17g}{337l}}$

The energy equation now includes the K.E. at the top.

From **a** $17mgl = \frac{337ml^2}{48}\Omega^2$

Any equivalent form is acceptable.

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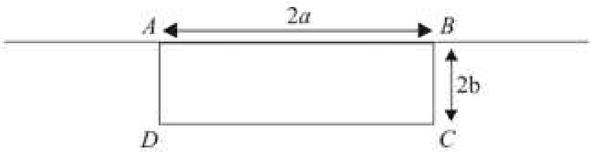
Exercise A, Question 11

Question:



The diagram shows a sign which hangs outside a shop. The sign is a thin rectangular metal plate which is free to rotate about a fixed smooth horizontal axis which lies along the side AB. The lengths of AB and BC are $2a$ and $2b$ respectively. The sign can be modelled as a uniform rectangular lamina. The sign is hanging freely below the axis when it receives a blow and starts to rotate with angular speed $k\sqrt{\frac{g}{b}}$.

a Find the least value of k for which the sign makes complete revolutions.

b If $k = 1.5$, find the angle BC makes with the upward vertical when the sign first comes to rest.

Solution:

a M.I. of sign about axis along $AB = \frac{4}{3}mb^2$

From the formula book, letting the mass of the rectangle be m.

For the least initial angular speed for complete revolutions:

$$\frac{1}{2}\left(\frac{4}{3}mb^2\right)\left(k\sqrt{\frac{g}{b}}\right)^2 = mg \times 2b$$

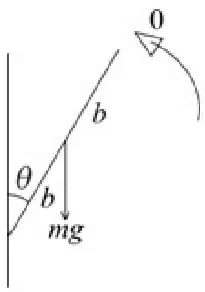
The angular speed at the top will be zero.

$$\frac{2}{3}mb^2 \times k^2\frac{g}{b} = 2mgb$$

$$k^2 = 3$$

The least value of k is $\sqrt{3}$

b



$$\text{Initial K.E.} = \frac{1}{2} \times \frac{4}{3}mb^2 \times \left(1.5\sqrt{\frac{g}{b}}\right)^2$$

$k = 1.5$

$$= 1.5mgb$$

$$\therefore mgb(1+\cos\theta) = 1.5\,mgb$$

P.E. gained = K.E. lost

$$1+\cos\theta = 1.5$$

$$\cos\theta = 0.5$$

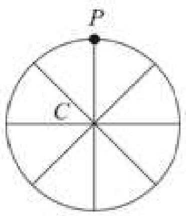
$$\theta = 60^\circ$$

$\therefore$ BC makes an angle of 60° with the upward vertical.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

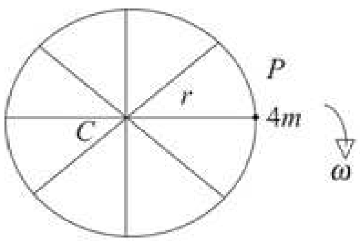
Exercise A, Question 12

Question:



A flywheel is made from a circular hoop of mass $6m$ and radius r and four equally spaced rods, each of mass m and length $2r$. A particle P of mass $4m$ is attached to the hoop at the end of one rod. The loaded flywheel is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the plane of the hoop through its centre, C. Initially the flywheel is at rest with P vertically above C, as shown in the diagram. The wheel is then slightly disturbed and begins to rotate. Find, in terms of r and g, the angular speed of the flywheel when PC is horizontal.

Solution:



M.I. of flywheel and particle about given axis through C

$$= 6mr^2 + 4 \times \frac{1}{3}mr^2 + 4mr^2$$

The flywheel is a hoop and 4 rods.

$$= \frac{34}{3}mr^2$$

$$\frac{1}{2}I\omega^2 = 4mgr$$

K.E. gained = P.E. lost

$$\frac{1}{2} \times \frac{34}{3}mr^2\omega^2 = 4mgr$$

$$\omega^2 = 4g \times \frac{6}{34r}$$

$$\omega = 2\sqrt{\frac{3g}{17r}}$$

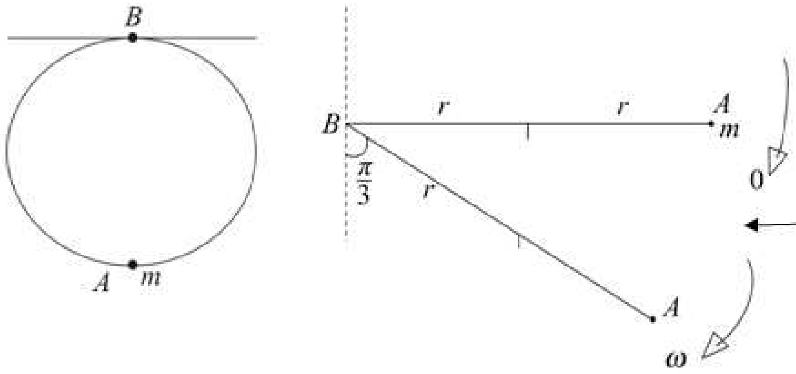
Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 13

Question:

A ring of mass $3m$ and radius r has a particle of mass m attached to it at the point A. The ring can rotate about a fixed smooth horizontal axis in the plane of the ring. The axis is tangential to the ring at the point B where AB is a diameter. The system is released from rest with AB horizontal. Find the angular speed of the ring when AB makes an angle $\frac{\pi}{3}$ with the downward vertical.

Solution:



The right-hand diagram is the side view.

M.I. of ring about perpendicular axis through centre $= 3mr^2$ ← From the formula book.

M.I. of ring about axis along a diameter $= \frac{1}{2} \times 3mr^2$ ← Perpendicular axes theorem.

M.I. of ring about tangential axis $= \frac{3}{2}mr^2 + 3mr^2$ ← Parallel axes theorem.

$\therefore$ M.I. of ring and particle about the given

$$\text{axis} = \frac{9}{2}mr^2 + m(2r)^2 = \frac{17}{2}mr^2$$

$$\frac{1}{2}I\omega^2 = 3mgr\cos\frac{\pi}{3} + mg(2r)\cos\frac{\pi}{3}$$ ← K.E. gained = P.E. lost

$$\frac{17}{4}mr^2\omega^2 = 3mgr \times \frac{1}{2} + 2mgr \times \frac{1}{2}$$

$$\frac{17}{4}r\omega^2 = \frac{5g}{2}$$

$$\omega^2 = \frac{10g}{17r}$$

$$\omega = \sqrt{\frac{10g}{17r}}$$

Solutionbank M5
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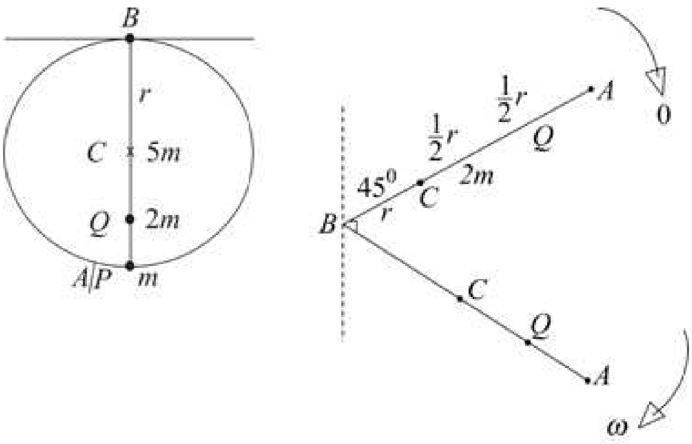
Exercise A, Question 14

Question:

A uniform circular disc has mass $5m$ and radius r. A particle P of mass m is attached to the disc at point A of its circumference. The centre of the disc is C. A second particle Q of mass $2m$ is attached to the disc at the mid-point of AC. The disc is free to rotate about a fixed smooth horizontal axis in the plane of the disc. The axis is tangential to the disc at point B, where AB is a diameter. The disc is released from rest with AB at an angle $45°$ with the upward vertical. When AB is at an angle $45°$ with the downward vertical the angular speed of the disc is ω. Show that $\omega^2 = \frac{80g\sqrt{2}}{59r}$.

Solution:



M.I. of disc about perpendicular axis through C

$= \frac{1}{2} \times 5mr^2$ ← From the formula book.

M.I. of disc about a diameter $= \frac{1}{4} \times 5mr^2$ ← Perpendicular axes theorem.

M.I. of disc about a tangential axis

$= \frac{5}{4}mr^2 + 5mr^2 = \frac{25}{4}mr^2$ ← Parallel axes theorem.

M.I. of loaded disc about the given axis

$$= \frac{25mr^2}{4} + 2m\left(\frac{3r}{2}\right)^2 + m(2r)^2$$

$$= \frac{59mr^2}{4}$$

$$\text{P.E. lost} = 5mgr\sqrt{2} + 2mg \times \frac{3r}{2}\sqrt{2} + mg \times 2r\sqrt{2}$$

$$= 10mgr\sqrt{2}$$

$$\therefore \frac{1}{2} \times \frac{59mr^2}{4}\omega^2 = 10mgr\sqrt{2}$$ ← K.E. gained = P.E. lost

$$\omega^2 = \frac{80g\sqrt{2}}{59r}$$

Solutionbank M5
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Exercise B, Question 1

Question:

A uniform rod of length $4a$ and mass m is free to rotate in a horizontal plane about a fixed smooth vertical axis through its centre. A horizontal force of constant magnitude is applied to a free end of the rod in a direction perpendicular to the rod. The rod rotates with angular acceleration $12\frac{g}{a}$. Find the magnitude of the force.

Solution:

$$\text{M.I. of rod} = \frac{1}{3}m \times (2a)^2 = \frac{4ma^2}{3}$$

From the formula book.

$$L = I\ddot{\theta}$$

$$F \times 2a = \frac{4ma^2}{3} \times \frac{12g}{a}$$

$$F = 8mg$$

L is the moment of the force about the axis.

The force has magnitude $8mg$.

Solutionbank M5
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Exercise B, Question 2

Question:

A uniform disc of radius 0.5 m and mass 2.4 kg is free to rotate in a horizontal plane about a fixed smooth vertical axis through its centre. A horizontal force of constant magnitude 10 N is applied at point A on the rim of the disc in the direction of the tangent to the disc at A.

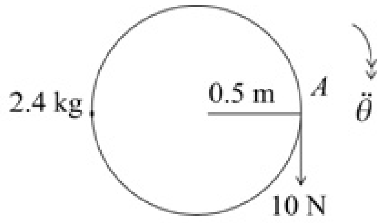
a Calculate the angular acceleration of the disc.

The disc starts from rest at time $t = 0$. Calculate

b the angular speed when $t = 2$,

c the angle the disc turns through in the first 2 s of the motion.

Solution:



a M.I. of disc $= \frac{1}{2} \times 2.4 \times 0.5^2$ ← From the formula book.

$= 0.3 \text{ kg m}^2$

$10 \times 0.5 = 0.3\ddot{\theta}$ ← Using $L = I\ddot{\theta}$

$\ddot{\theta} = \frac{10 \times 0.5}{0.3} = \frac{50}{3}$

The angular acceleration is $16\frac{2}{3} \text{ rad s}^{-2}$.

b $t = 2$ $\omega_1 = \omega_0 + \alpha t$

$\omega_0 = 0$

$\alpha = \frac{50}{3}$ $\omega_1 = 0 + \frac{50}{3} \times 2 = \frac{100}{3}$

The angular speed is $33\frac{1}{3} \text{ rad s}^{-1}$.

c $t = 2$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega_0 = 0$ $\theta = 0 + \frac{1}{2} \times \frac{50}{3} \times 2^2$

$\alpha = \frac{50}{3}$ $\theta = \frac{100}{3}$

The disc has turned through $33\frac{1}{3}$ rad.

Solutionbank M5

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

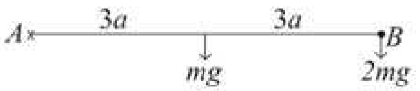
Question:

A uniform rod AB of mass m and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB at A. A particle of mass $2m$ is attached to the rod at B. The loaded rod is released from rest with AB horizontal. Find

a the initial angular acceleration of the rod,

b the angular acceleration when AB makes an angle $\frac{\pi}{3}$ with the downward vertical.

Solution:



a M.I. of rod and particle about the given axis through $A = \frac{4}{3}m\times(3a)^2 + 2m\times(6a)^2$

$$= 84ma^2$$

When the rod is released:

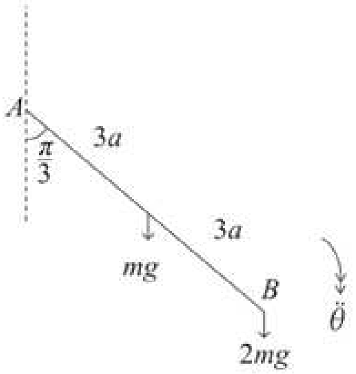
$$mg\times 3a + 2mg\times 6a = 84ma^2\ddot{\theta}$$

Using $L = I\ddot{\theta}$

$$\ddot{\theta} = \frac{15g}{84a} = \frac{5g}{28a}$$

The initial angular acceleration is $\frac{5g}{28a}$.

b



$$mg\times 3a\sin\frac{\pi}{3} + 2mg\times 6a\sin\frac{\pi}{3} = 84ma^2\ddot{\theta}$$

Using $L = I\ddot{\theta}$

$$15g\frac{\sqrt{3}}{2} = 84a\ddot{\theta}$$

$$\ddot{\theta} = \frac{15g}{84a}\times\frac{\sqrt{3}}{2} = \frac{5g\sqrt{3}}{56}$$

The angular acceleration is $\frac{5g\sqrt{3}}{56a}$.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

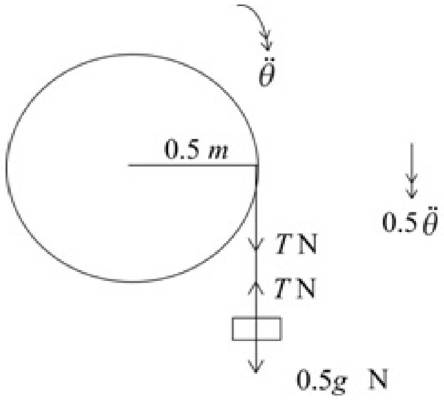
Question:

A pulley wheel of mass 2 kg and radius 0.5 m has one end of a rope attached to a point of the rim of the wheel. The rope is wound several times around the wheel. A fixed smooth horizontal axis passes through the centre of the wheel. A brick of mass 0.5 kg is attached to the free end of the rope. Initially the system is held at rest with the brick hanging freely with the rope taut. The system is then released and the wheel begins to rotate in a vertical plane perpendicular to the axis. The pulley wheel can be modelled as a uniform circular disc, the rope as a light inextensible string and the brick as a particle. Calculate

a the tension in the rope,

b the distance the brick falls in the first second after the system is released.

Solution:



The (linear) acceleration of the brick is $r\ddot{\theta}$.

a For the brick:

$0.5g - T = 0.5 \times 0.5\ddot{\theta}$ ①

Using $F = ma$ with $m = 0.5$ kg and $a = 0.5\ddot{\theta}$

For the wheel:

$\text{M.I.} = \frac{1}{2} \times 2 \times 0.5^2 = 0.25 \text{ kg m}^2$

$\therefore T \times 0.5 = 0.25\ddot{\theta}$

$T = 0.5\ddot{\theta}$ ②

Using $L = I\ddot{\theta}$

Substitute in ①:

$0.5g - T = 0.5T$

$\therefore T = \frac{1}{3}g$

The tension is $\frac{1}{3}$g N (or 3.27 N)

b From ② $\ddot{\theta} = 2T = \frac{2}{3}g$

For the brick:

$a = 0.5\ddot{\theta} = \frac{1}{3}g$

Find the angular acceleration so you can use the constant acceleration equations.

$u = 0$

$t = 1$

$s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times \frac{g}{3} \times 1^2$

$s = 1.633$

The brick falls 1.63 m (3 s.f.) in the first second.

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Exercise B, Question 5

Question:

A uniform rod AB of mass m and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB through the point D of the rod where $AD = a$. The rod is released from rest with AB horizontal. Calculate the magnitude of the force exerted on the axis

a when AB is vertical with A above D

b when AB makes an angle of $45°$ with the downward vertical.

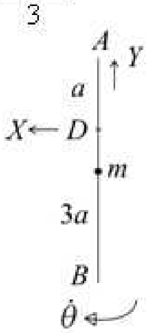
Solution:

M.I. of rod about given axis through D

$= \frac{1}{3}m \times (2a)^2 + ma^2$ ← Using the parallel axes theorem.

$= \frac{7ma^2}{3}$

a



← You must find $\dot{\theta}$ and $\ddot{\theta}$ when the rod is vertical.

$\frac{1}{2}I\dot{\theta}^2 = mga$

$\frac{1}{2} \times \frac{7ma^2}{3}\dot{\theta}^2 = mga$ ← K.E. gained = P.E. lost

$\dot{\theta}^2 = \frac{6g}{7a}$

$\ddot{\theta} = 0$ (from $L = I\ddot{\theta}$) ← When the rod is vertical there is no force with a non-zero moment about the axis.

Consider the motion of a particle of mass m at the centre of mass of the rod.

$Y - mg = ma\dot{\theta}^2$

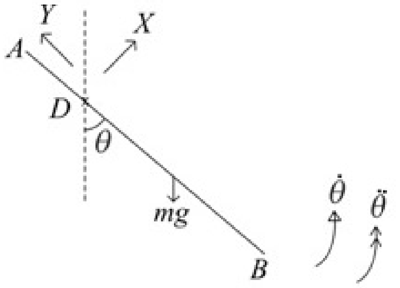
$Y = mg + ma \times \frac{6g}{7a}$ ← The particle is moving in a circle, radius a.

$Y = \frac{13}{7}mg$

$\ddot{\theta} = 0 \Rightarrow X = 0$

$\therefore$ Magnitude is $\frac{13mg}{7}$

b



$\dot{\theta}$ and $\ddot{\theta}$ are in the direction of increasing θ.

At θ to the downward vertical:

$$\frac{1}{2}\times\frac{7ma^2}{3}\dot{\theta}^2 = mga\cos\theta$$

K.E. gained = P.E. last

$$\dot{\theta}^2 = \frac{6g}{7a}\cos\theta$$

When $\theta = 45°$ $\quad \dot{\theta}^2 = \frac{6g}{7a}\times\frac{1}{\sqrt{2}}$

$$2\ddot{\theta} = -\frac{6g}{7a}\sin\theta$$

You can differentiate $\dot{\theta}^2$ with respect to θ to obtain $\ddot{\theta}$. Alternatively, you can use $L = I\ddot{\theta}$

When $\theta = 45°$ $\ddot{\theta} = -\frac{3g}{7a}\times\frac{1}{\sqrt{2}}$

For the particle at the centre of mass of the rod:
Parallel to the rod:

$$Y - mg\cos\theta = ma\dot{\theta}^2$$

$$Y = mg\times\frac{1}{\sqrt{2}} + ma\times\frac{6g}{7a}\times\frac{1}{\sqrt{2}}$$

$$Y = \frac{13mg}{7\sqrt{2}}$$

Perpendicular to the rod:

$$X - mg\sin\theta = ma\ddot{\theta}$$

$$X = mg\times\frac{1}{\sqrt{2}} - ma\times\frac{3g}{7a\sqrt{2}}$$

$$X = \frac{4mg}{7\sqrt{2}}$$

$\therefore$ Magnitude of the force $= \sqrt{(X^2 + Y^2)}$

The magnitude is the same for the force on the axis and the force on the particle.

$$= \frac{mg}{7\sqrt{2}}\sqrt{\left(13^2 + 4^2\right)}$$

$$= \frac{mg}{7}\sqrt{\frac{185}{2}}$$

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Exercise B, Question 6

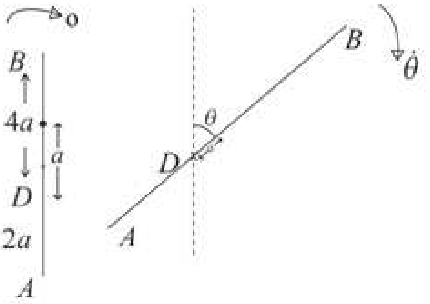
Question:

A uniform rod AB of mass m and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to AB through the point D of the rod where $AD = 2a$. The rod is initially at rest with A vertically below D but is then slightly disturbed and starts to rotate. Find

a the angular speed when AB has turned through an angle θ ,

b the magnitude of the force on the axis when the rod is vertical with B below D.

Solution:



a M.I. of rod about given axis through D

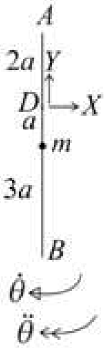
$= \frac{1}{3}m \times (3a)^2 + ma^2 = 4ma^2$ — Use the parallel axes theorem.

$\frac{1}{2}I\dot{\theta}^2 = mga(1-\cos\theta)$ — K.E. gained = P.E. lost

$2ma^2\dot{\theta}^2 = mga(1-\cos\theta)$

$\dot{\theta} = \sqrt{\left[\frac{g}{2a}(1-\cos\theta\right]}$

b



There is no horizontal force other than $x \Rightarrow \ddot{\theta} = 0$ — Using $L = I\ddot{\theta}$

Consider the motion of a particle of mass m at the centre of mass of the rod: — Alternatively, when B is vertically below D the angular speed is maximum, so $\ddot{\theta} = 0$

Along the rod: $Y - mg = ma\dot{\theta}^2$

$Y - mg = ma \times \frac{g}{2a}[1-(-1)]$ — $\theta = 180° \Rightarrow \cos\theta = -1$

$Y - mg = mg$

$Y = 2mg$

Perpendicular to the rod:

$\ddot{\theta} = 0$

$\therefore X = 0$

The magnitude of the force is $2mg$. — The magnitude is the same for the force on the axis and the force on the particle.

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Exercise B, Question 7

Question:

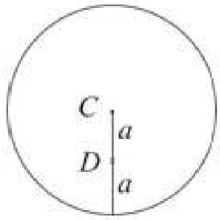
A uniform circular disc of mass m and radius $2a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the disc through a point, D, which is at a distance a from the centre of the disc, C. The disc is initially at rest with C vertically above D. The disc is then slightly disturbed and begins to rotate. Find the magnitude of the force on the axis

a when CD is horizontal

b when CD is vertical with C below D.

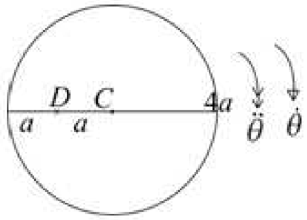
Solution:

a



M.I. of disc about axis through $C = \frac{1}{2}m(2a)^2 = 2ma^2$ ← From the formula book.

M.I. of disc about axis through $D = 2ma^2 + ma^2$
$= 3ma^2$ ← By the parallel axes theorem.



Energy from rest to CD horizontal

$\frac{1}{2}I\dot{\theta}^2 = mga$ ← K.E. gained = P.E. lost

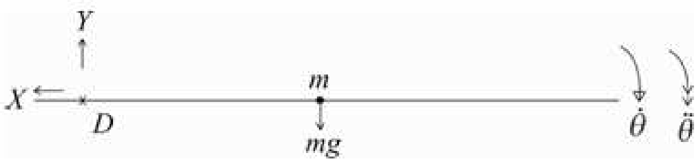
$\frac{3ma^2}{2}\dot{\theta}^2 = mga$

$\dot{\theta}^2 = \frac{2g}{3a}$

$L = I\ddot{\theta}$ ← Equation of rotational motion

$mga = 3ma^2\ddot{\theta}$

$\ddot{\theta} = \frac{g}{3a}$



Consider the motion of a particle of mass m at the centre of mass of the disc:

Horizontally:

$X = ma\dot{\theta}^2$ ← Use $F = ma$

$X = ma \times \frac{2g}{3a}$

$X = \frac{2mg}{3}$

Vertically:

$$mg - Y = ma\ddot{\theta}$$

$$Y = mg - ma \times \frac{g}{3a}$$

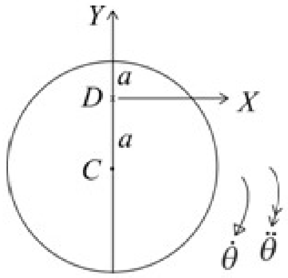
$$Y = \frac{2mg}{3}$$

$$\therefore \text{ Magnitude} = \sqrt{\left(\frac{2mg}{3}\right)^2 + \left(\frac{2mg}{3}\right)^2}$$

$$= \frac{2mg}{3}\sqrt{2}$$

Use $F = ma$

b



Energy from rest to CD vertical:

$$\frac{1}{2}I\dot{\theta}^2 = 2mga$$

$$\frac{3ma^2}{2}\dot{\theta}^2 = 2mga$$

$$\dot{\theta}^2 = \frac{4g}{3a}$$

K.E. gained = PE lost

No horizontal force apart from $x \Rightarrow \ddot{\theta} = 0$

Use the equation of rotational motion

For a particle of mass m at C

Vertically:

$$Y - mg = ma\dot{\theta}^2$$

$$Y = mg + \frac{4mg}{3} = \frac{7mg}{3}$$

Use $F = ma$

Horizontally: $X = -ma\ddot{\theta} = 0$

Use $F = ma$

$\therefore$ Magnitude is $\frac{7mg}{3}$

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 8

Question:

A uniform rod AB of mass m and length $2a$ is attached to a fixed smooth hinge at A. The rod is released from rest from a horizontal position and rotates in a vertical plane perpendicular to the hinge.

a Show that, when AB has rotated through an angle θ

$$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

When AB has rotated through an angle θ, the force exerted by AB on the axis is F.

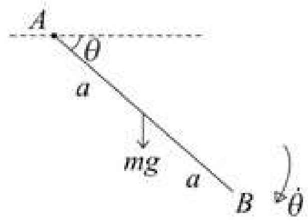
b Find the magnitudes of the components, parallel and perpendicular to AB, of F.

c Show that the horizontal component of F is greatest when $\theta = \frac{\pi}{4}$.

d Find the vertical component of F when $\theta = \frac{\pi}{4}$.

Solution:

a



M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

Energy:

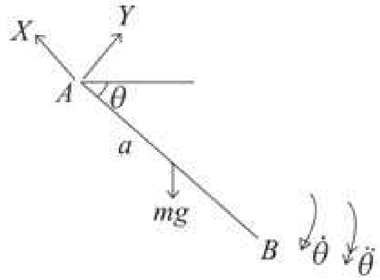
$$\frac{1}{2}\times\left(\frac{4}{3}ma^2\right)\dot{\theta}^2 = mga\sin\theta$$

← The rod starts from rest with AB horizontal.

$$2a\dot{\theta}^2 = 3g\sin\theta$$

$$\text{or } 2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

b



← Consider the motion of a particle of mass m at the centre of mass of the rod.

For the component parallel to AB:

$$X - mg\sin\theta = ma\dot{\theta}^2$$

← Use $F = ma$

$$X = mg\sin\theta + m\times\frac{3g}{2}\sin\theta$$

← Use the result from **a**.

$$X = \frac{5mg}{2}\sin\theta$$

For the component perpendicular to AB:

$$mg\cos\theta - Y = ma\ddot{\theta}$$

← Use $F = ma$

$$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

$$4a\frac{d^2\theta}{dt^2} = 3g\cos\theta$$

← Differentiate the result from **a** with respect to θ

$$a\ddot{\theta} = \frac{3g}{4}\cos\theta$$

$$\therefore Y = mg\cos\theta - \frac{3}{4}mg\cos\theta = \frac{1}{4}mg\cos\theta$$

c Horizontal component

$= X\cos\theta - Y\sin\theta$

$= \frac{5}{2}mg\sin\theta\cos\theta - \frac{1}{4}mg\sin\theta\cos\theta$

$= \frac{9mg}{4}\sin\theta\cos\theta$

$= \frac{9}{8}mg\sin 2\theta$

You can differentiate this to obtain the maximum but the trigonometric method is much simpler!

$\therefore$ Horizontal component is maximum when $\sin 2\theta = 1$

$\theta = \frac{\pi}{4}$

$\therefore$ Maximum when $\theta = \frac{\pi}{4}$

d Vertical component

$= X\sin\theta + Y\cos\theta$

$= \frac{5}{2}mg\sin^2\theta + \frac{1}{4}mg\cos^2\theta$

$\theta = \frac{\pi}{4}$

Vertical component

$= \frac{5}{2}mg\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{4}mg\left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{5mg}{4} + \frac{1}{8}mg$

$= \frac{11}{8}mg$

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Exercise B, Question 9

Question:

A uniform wire of mass m and length $6a$ is bent to form a rectangle $ABCD$ with $AB = 2a$. It is hung with corner A over a fixed smooth horizontal nail. Initially it is held at rest with AB horizontal and D below A. The plane of the rectangle is perpendicular to the nail.

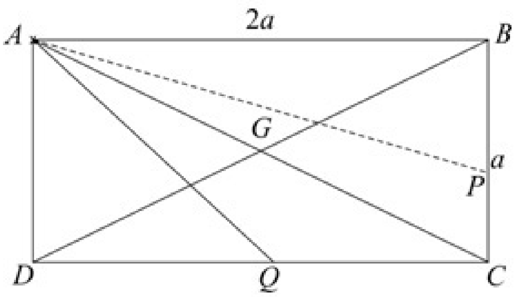
a Show that the moment of inertia of the framework about the nail is $2ma^2$.

b Show that the angular speed $\dot{\theta}$ of the wire when AC is vertical is given by $\dot{\theta}^2 = \frac{g}{2a}(\sqrt{5}-1)$.

c Find the magnitude of the resultant force on the nail when AC is vertical.

Solution:

a



G is the centre of mass of the framework.

P and Q are the mid-points of BC and CD respectively.

M.I. of rectangle about nail

$$= \frac{4}{3}\times\left(\frac{1}{3}m\right)\times a^2 + \left\{\frac{1}{3}\times\frac{1}{6}m\left(\frac{1}{2}a\right)^2 + \frac{1}{6}m\left(4a^2+\frac{1}{4}a^2\right)\right\}$$
$$+\left\{\frac{1}{3}\times\left(\frac{1}{3}m\right)\times a^2 + \frac{1}{3}m\times 2a^2\right\} + \frac{4}{3}\times\left(\frac{1}{6}m\right)\left(\frac{a}{2}\right)^2$$
$$= \frac{4ma^2}{9} + \frac{ma^2}{72} + \frac{17ma^2}{24} + \frac{ma^2}{9} + \frac{2ma^2}{3} + \frac{ma^2}{18}$$
$$= 2ma^2$$

You must work from the centres of mass when using the parallel axes theorem for BC and CD.

$AP^2 = 4a^2 + \frac{1}{4}a^2$

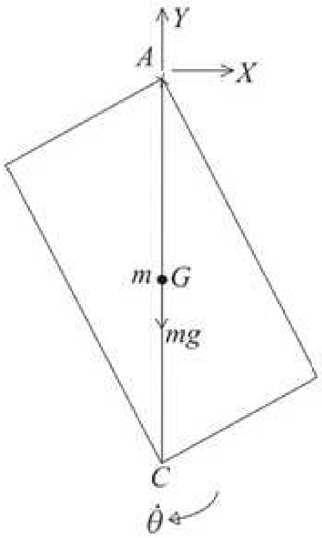
$AQ^2 = a^2 + a^2$

b Energy:

$$\frac{1}{2}\times 2ma^2\dot{\theta}^2 = mg\left(\frac{a}{2}\sqrt{5} - \frac{a}{2}\right)$$
$$2a\dot{\theta}^2 = g(\sqrt{5}-1)$$
$$\dot{\theta}^2 = \frac{g}{2a}(\sqrt{5}-1)$$

$AG^2 = a^2 + \frac{1}{4}a^2 = \frac{5a^2}{4}$

c



Consider the motion of a particle of mass m placed at the centre of mass of the framework.

Vertically:

$Y - mg = m \times AG \times \dot{\theta}^2$

$Y - mg = \frac{ma}{2}\sqrt{5} \times \frac{g}{2a}(\sqrt{5} - 1)$

$AG = \frac{a}{2}\sqrt{5}$ (from **a**)

$Y = mg + \frac{mg}{4} \times 5 - \frac{mg}{4}\sqrt{5}$

$= \frac{9mg}{4} - \frac{mg\sqrt{5}}{4}$

Horizontally:

$-X = \frac{ma}{2}\sqrt{5}\ddot{\theta}$

$\dot{\theta}$ is maximum when AC is vertical

$\Rightarrow \ddot{\theta} = 0$

$\therefore X = 0$

Or use $L = I\ddot{\theta}$ with $L = 0$

$\therefore$ The magnitude of the resultant force on the nail is $\frac{9mg}{4} - mg\frac{\sqrt{5}}{4}$ or

$\frac{mg}{4}(9 - \sqrt{5})$

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Exercise B, Question 10

Question:

A uniform square lamina $ABCD$ of mass m and side $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through A perpendicular to $ABCD$. The lamina is hanging in equilibrium with C below A when it receives an impulse and begins to rotate with angular speed $\sqrt{\frac{3g}{a}}$.

a Show that the lamina will perform complete revolutions.

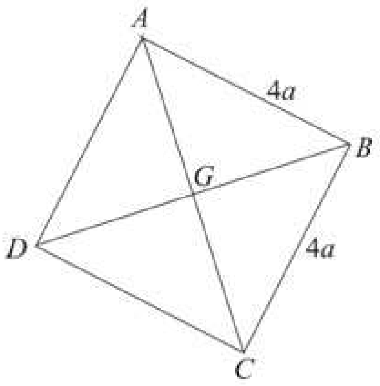
b Find the magnitude of the horizontal and vertical components of the force on the axis

i when C is vertically above A,

ii when AC is horizontal.

Solution:

a



G is the centre of mass of $ABCD$.
$AG^2 = (2a)^2 + (2a)^2$
$= 8a^2$

M.I. of lamina about axis through $A = \frac{1}{3}m\left(4a^2 + 4a^2\right) + m \times 8a^2$

$= \frac{32ma^2}{3}$

Energy:

$$\frac{1}{2} \times \frac{32ma^2}{3} \times \frac{3g}{a} - \frac{1}{2} \times \frac{32ma^2}{3}\omega^2 = mg \times 2a\sqrt{8}$$

$$16g - \frac{16a\omega^2}{3} = 4g\sqrt{2}$$

ω is the angular speed when C is vertically above A.

For complete revolutions $\omega^2 \geq 0$

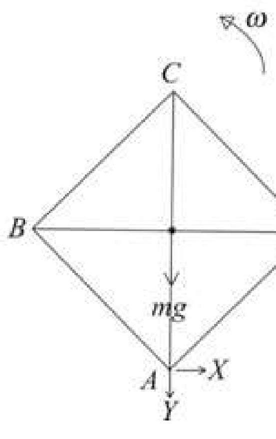
$$\frac{16a\omega^2}{3} = 16g - 4g\sqrt{2}$$

$$a\omega^2 = \frac{3}{4}g(4 - \sqrt{2}) > 0$$

$\therefore$ The lamina will perform complete revolutions.

As long as it is clear that $\omega^2 > 0$ there is no need to evaluate $\frac{3}{4}g(4 - \sqrt{2})$

b i



Consider the motion of a particle of mass m at the centre of mass of the lamina.

When C is vertically above A

$$\omega^2 = \frac{3g}{4a}\left(4 - \sqrt{2}\right)$$

From **a**.

Vertically:

$mg + Y = m \times 2a\sqrt{2}\omega^2$

$Y = 2ma\sqrt{2} \times \frac{3g}{4a}(4 - \sqrt{2}) - mg$

$Y = 6mg\sqrt{2} - 4mg$

Use $F = ma$

Horizontally:
when AC is vertical angular speed is a minimum.

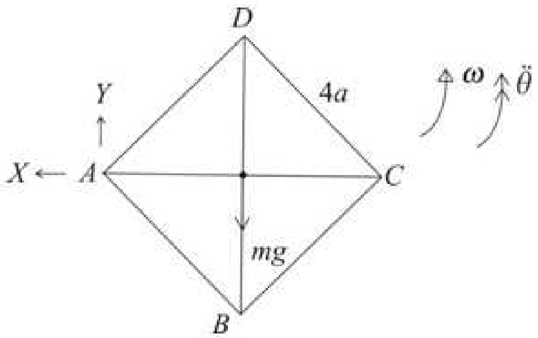
$\therefore \ddot{\theta} = 0$

$\therefore$ horizontal component of force $= 0$

The horizontal component is zero and the vertical component is $2mg(3\sqrt{2} - 2)$.

Or you can use $L = I\ddot{\theta}$ with $L = 0$

ii



Energy (to AC being horizontal):

$$\frac{1}{2} \times \frac{32ma^2}{3} \times \frac{3g}{a} - \frac{1}{2} \times \frac{32ma^2}{3}\omega_1^2 = mg \times a\sqrt{8}$$

$$16g - \frac{16}{3}a\omega_1^2 = g\sqrt{8}$$

$$\omega_1^2 = \frac{3g}{16a}(16 - \sqrt{8})$$

Horizontally:

$X = ma\sqrt{8} \times \frac{3g}{16a}(16 - \sqrt{8})$

$= \frac{3mg}{2}(4\sqrt{2} - 1)$

Use $F = ma$.

Any equivalent form is acceptable.

Vertically:

$Y - mg = ma\sqrt{8}\ddot{\theta}$

Use $F = ma$.

$mga\sqrt{8} = -\frac{32ma^2}{3}\ddot{\theta}$

Use $L = I\ddot{\theta}$.

$\therefore ma\ddot{\theta} = -mg\sqrt{8} \times \frac{3}{32}$

$\therefore Y = mg - mg\sqrt{8} \times \frac{3}{32}\sqrt{8} = \frac{mg}{4}$

The horizontal component has magnitude $\frac{3mg}{2}(4\sqrt{2} - 1)$ and the vertical component has magnitude $\frac{1}{4}mg$.

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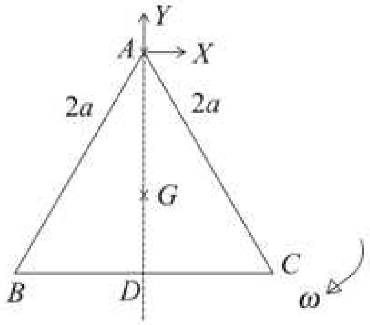
Exercise B, Question 11

Question:

Three equal uniform rods, each of mass m and length $2a$, are joined to form an equilateral triangle ABC. The triangular frame can rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to ABC through A. The mid-point of BC is D. The frame is released from rest with AD horizontal and C below AB. Find the magnitude of the force on the axis when AD is vertical.

[You may assume that the centre of mass of the triangle is at G where G divides AD in the ratio 2 : 1.]

Solution:



G is the centre of mass of the framework $AG:GD = 2:1$

M.I. of triangle about axis through A

$= 2\times\frac{4}{3}ma^2 + \frac{1}{3}ma^2 + m\times 3a^2$

$= \frac{8ma^2}{3} + \frac{ma^2}{3} + 3ma^2$

$= 6ma^2$

$AD^2 = 4a^2 - a^2$
$= 3a^2$

From the formula book and using the parallel axes theorem.

Energy:

$\frac{1}{2}\times 6ma^2\omega^2 = 3mg\times\frac{2}{3}\times a\sqrt{3}$

$ma\omega^2 = \frac{2mg}{3}\sqrt{3}$

AD was horizontal at the start.

When AD is vertical:

vertically:

$Y - 3mg = 3m\times\frac{2a}{3}\sqrt{3}\omega^2$

$Y = 3mg + 2\sqrt{3}\times\frac{2}{3}mg\sqrt{3}$

$Y = 7mg$

Consider the motion of a particle of mass $3m$ placed at G.

Use $F = ma$.

$AG = \frac{2}{3}AD = \frac{2a}{3}\sqrt{3}$

When AD is vertical, angular speed is maximum.

$\therefore \ddot{\theta} = 0$

$\therefore$ no horizontal component

$\therefore$ The force on the axis has magnitude $7mg$.

Or use $L = I\ddot{\theta}$ with $L = 0$

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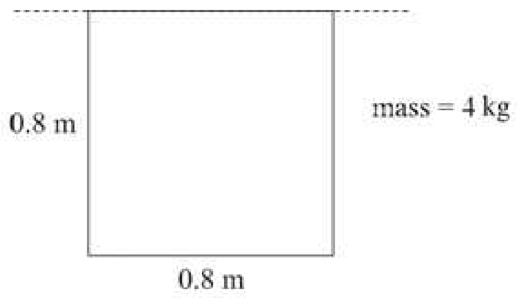
Exercise C, Question 1

Question:

A uniform square lamina of side 0.8 m and mass 4 kg is free to rotate about a fixed smooth axis which coincides with one of its sides. Calculate the gain of angular momentum when the angular speed of the lamina is increased from 2 rad s^{-1} to 5 rad s^{-1}.

Solution:



M.I. of lamina about given axis $= \frac{4}{3} \times 4 \times 0.4^2$ ← From the formula book.

Gain in angular momentum

$$= I\omega_1 - I\omega_0$$
$$= \frac{4}{3} \times 4 \times 0.4^2 (5-2)$$
$$= 2.56 \text{ Nms}$$

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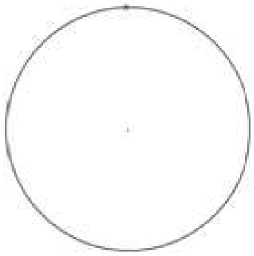
Exercise C, Question 2

Question:

A uniform hoop of mass 1.2 kg and radius 1.5 m is rotating at a constant angular speed of 6 rad s^{-1} about a fixed smooth horizontal axis through a point of the circumference of the hoop, perpendicular to the plane of the hoop. Calculate the angular momentum of the hoop.

Solution:



M.I. of hoop about perpendicular axis through its centre $= mr^2$ ← From the formula book.

$= 1.2 \times 1.5^2$ kg m^2

M.I. of hoop about given axis $= 1.2 \times 1.5^2 + 1.2 \times 1.5^2$ ← Use the parallel axes theorem.

$= 5.4$ kgm^2

Angular momentum

$= 5.4 \times 6$

$= 32.4$ Nms ← Angular momentum $= I\omega$

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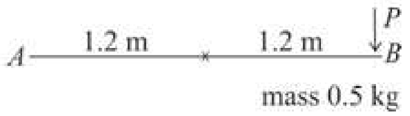
Exercise C, Question 3

Question:

A uniform rod AB of length 2.4 m and mass 0.5 kg is rotating in a horizontal plane at 6 rad s^{-1} about a fixed smooth vertical axis through its centre. A retarding force of constant magnitude P newtons is applied at B in a direction perpendicular to AB in the plane of the motion. The rod is brought to rest in 5 seconds. Calculate the value of P.

Solution:



M.I. of rod about vertical axis through centre $= \frac{1}{3}ml^2$ ← From the formula book.

$= \frac{1}{3} \times 0.5 \times 1.2^2 \text{kgm}^2$

Angular momentum lost

$= \frac{1}{3} \times 0.5 \times 1.2^2 \times 6 \text{Nms}$

$\therefore P \times 5 \times 1.2 = \frac{1}{3} \times 0.5 \times 1.2^2 \times 6$

$P = \frac{1}{3} \times \frac{0.5 \times 1.2^2 \times 6}{5 \times 1.2}$

$P = 0.24$

Moment of impulse = change in angular momentum

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Exercise C, Question 4

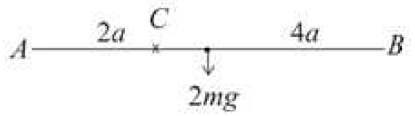
Question:

A uniform rod AB of mass $2m$ and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through the point C of the rod where $AC = 2a$. The rod is released from rest with AB horizontal. When the rod is vertical with B below C, the end B strikes a stationary particle of mass m. The particle adheres to the rod.

a Show that the angular speed of the rod immediately after the impact is $\frac{1}{3}\sqrt{\frac{g}{2a}}$.

b Calculate the angle between the rod and the downward vertical when the rod first comes to instantaneous rest.

Solution:



M.I. of rod about horizontal axis through C

$= \frac{1}{3}(2m)\times(3a)^2 + 2ma^2$

$= 8ma^2$

From the formula book and using the parallel axes theorem.

a Energy from release to impact:

$\frac{1}{2}I\dot{\theta}^2 = 2mga$

K.E. gained = P.E. lost

$4ma^2\dot{\theta}^2 = 2mga$

$\dot{\theta}^2 = \frac{2mga}{4ma^2} = \frac{g}{2a}$

For the impact:

$I\dot{\theta} = \left[I + m(4a)^2\right]\omega$

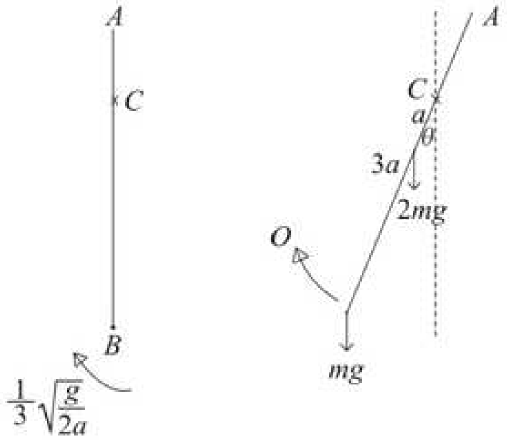
ω is the angular speed after the impact.

$8ma^2\sqrt{\frac{g}{2a}} = \left(8ma^2 + 16ma^2\right)\omega$

$8ma^2\sqrt{\frac{g}{2a}} = 24ma^2\omega$

$\omega = \frac{1}{3}\sqrt{\frac{g}{2a}}$

b



Energy to the highest point:

$$\frac{1}{2}I'\omega^2 = 2mg(a - a\cos\theta) + mg(4a - 4a\cos\theta)$$

I' is the M.I. of the rod and the particle.

$$\frac{1}{2}\times 24ma^2\left(\frac{1}{3}\sqrt{\frac{g}{2a}}\right)^2 = 6mga(1-\cos\theta)$$

$I' = 24ma^2$ from **a**

$$12a\times\frac{1}{9}\times\frac{g}{2a} = 6g(1-\cos\theta)$$

$$\frac{6}{9} = 6(1-\cos\theta)$$

$$\cos\theta = 1-\frac{1}{9}=\frac{8}{9}$$

$$\theta = 27.26\ldots$$

The angle between the rod and the downward vertical is $27.3°$ (3 s.f.)

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Exercise C, Question 5

Question:

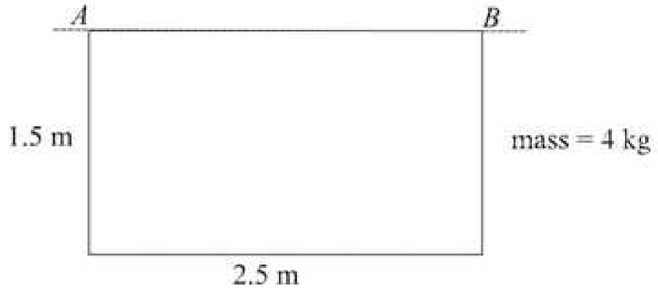
A rectangular sign is hanging outside a shop. The sign has mass 4 kg and measures 1.5 m by 2.5 m. It is free to rotate about a fixed smooth horizontal axis which coincides with a long side of the sign. The sign is hanging vertically at rest when it receives an impulse, perpendicular to its plane, at its centre of mass. The sign first comes to rest when it is horizontal. Calculate

a the initial angular speed of the sign,

b the magnitude of the impulse.

(You may assume that the sign can be modelled as a uniform rectangular lamina.)

Solution:



M.I. of sign about axis along $AB = \frac{4}{3} \times 4 \times \left(\frac{1.5}{2}\right)^2 = 3 \text{ kg m}^2$ ← From the formula book.

a Energy from just after the impact until the sign is horizontal.

$$\frac{1}{2} I\omega^2 = 4g \times 0.75$$ ← ω is the initial angular speed

$$\frac{1}{2} \times 3\omega^2 = 4g \times 0.75$$

$$\omega^2 = \frac{8g \times 0.75}{3}$$

$$\omega = 4.427\ldots$$

The initial angular speed is 4.4 rad s^{-1} (2 s.f.)

b For the impact:

Moment of impulse = change in angular momentum

$$J \times 0.75 = 3 \times 4.427$$ ← J is the magnitude of the impulse.

$$J = \frac{3 \times 4.427}{0.75}$$

$$J = 17.7\ldots$$

The magnitude of the impulse is 18 N (2 s.f.)

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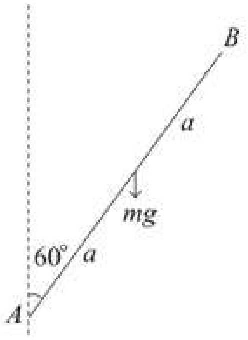
Exercise C, Question 6

Question:

A uniform rod AB of mass m and length $2a$ is freely hinged at A. The rod is released from rest with AB at $60°$ with the upward vertical through A. When AB is horizontal it hits a small fixed peg at point C where $AC = 1.5a$. The angular speed of the rod immediately after the impact is half its speed immediately before the impact. Find the impulse exerted by the peg on the rod.

Solution:



M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

Energy from release to horizontal:

$$\frac{1}{2}I\dot{\theta}^2 = mga\cos 60$$ ← gain of K.E. = loss of P.E.

$$\frac{2}{3}ma^2\dot{\theta}^2 = mga \times \frac{1}{2}$$

$$\dot{\theta}^2 = \frac{3g}{4a}$$

For the impact

$$1.5aJ = I\sqrt{\frac{3g}{4a}} + I \times \frac{1}{2}\sqrt{\frac{3g}{4a}}$$ ← J is the magnitude of the impulse. The direction of rotation is reversed.

$$1.5aJ = \left(\frac{4}{3}ma^2 + \frac{1}{2}\times\frac{4}{3}ma^2\right)\sqrt{\frac{3g}{4a}}$$

$$1.5a\,J = 2ma^2\sqrt{\frac{3g}{4a}}$$

$$J = \frac{2ma}{2}\sqrt{\frac{3g}{a}} \times \frac{1}{1.5} = \frac{2m}{3}\sqrt{3ga}$$

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Exercise C, Question 7

Question:

A uniform rod AB of mass m and length $2a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. When the rod is hanging at rest with B vertically below A, the end B receives an impulse of magnitude J in a direction perpendicular to the axis of rotation.

a Show that, for the rod to rotate in a complete circle,

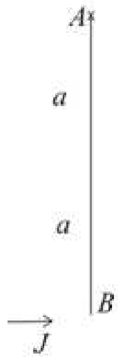
$$J \geq 2m\sqrt{\frac{ga}{3}}$$

Given that $J = \frac{2m}{3}\sqrt{\frac{ga}{3}}$

b find the angle the rod turns through before first coming to instantaneous rest.

Solution:

a



M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

For the impact:

$2aJ = I\omega$

$$\omega = 2aJ \times \frac{3}{4ma^2} = \frac{3J}{2ma}$$

Energy from impact to B vertically above A:

$$\frac{1}{2}I\omega^2 - \frac{1}{2}I\dot{\theta}^2 = mg \times 2a$$ ← Loss of K.E. = gain of P.E.

$$\frac{2ma^2}{3}\left(\frac{3J}{2ma}\right)^2 - \frac{2ma^2}{3}\dot{\theta}^2 = 2mga$$

For complete circles $\dot{\theta}^2 \geq 0$

$$\therefore \frac{2ma^2}{3} \times \frac{9J^2}{4m^2a^2} - 2mga \geq 0$$

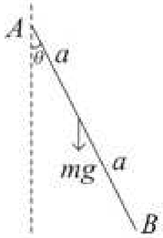
$$J^2 \geq 2mga \times \frac{2m^2a^2}{3ma^2}$$

$$J^2 \geq \frac{4m^2ga}{3}$$

$$J \geq 2m\sqrt{\frac{ga}{3}}$$

b $J = \frac{2m}{3}\sqrt{\frac{ga}{3}}$

$\therefore$ speed just after the impact $= \frac{3J}{2ma} = \frac{3}{2ma} \times \frac{2m}{3}\sqrt{\frac{ga}{3}} = \sqrt{\frac{g}{3a}}$ ← Use your result from **a**.



Energy from lowest to highest point:

$$\frac{1}{2}\left(\frac{4ma^2}{3}\right)\left(\sqrt{\frac{g}{3a}}\right)^2 = mga(1-\cos\theta)$$ ← K.E. lost = P.E. gained

$$\frac{2ma^2}{3} \times \frac{g}{3a} = mga(1-\cos\theta)$$

$$\frac{2}{9} = 1-\cos\theta$$

$$\cos\theta = \frac{7}{9}$$

$$\theta = 38.94\ldots$$

The rod turns through 38.9° (3 s.f.)

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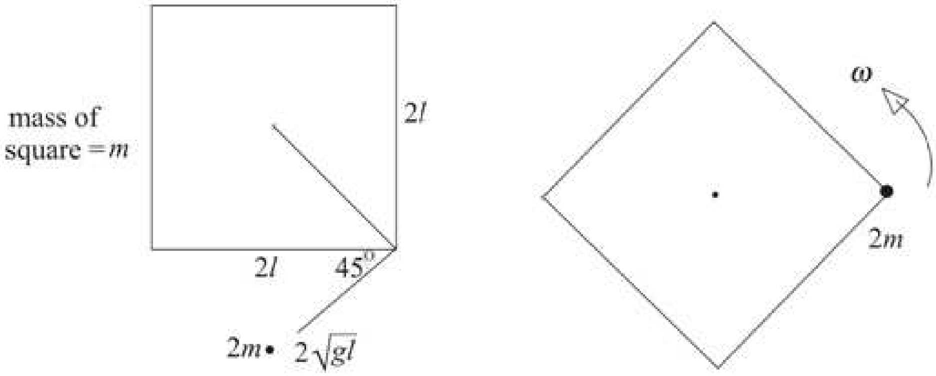
Exercise C, Question 8

Question:

A uniform square lamina of mass m and side $2l$ is free to rotate in a horizontal plane about a fixed smooth vertical axis through the centre of the lamina. Initially the lamina is at rest. A particle of mass $2m$ is moving in the plane of the lamina towards the lamina with speed $2\sqrt{gl}$ and in a direction at $45°$ to a side. The particle strikes and adheres to the lamina at a corner. Find the angular speed with which the lamina begins to turn.

Solution:



M.I. of square lamina about perpendicular axis through centre

$= \frac{1}{3}m(l^2 + l^2)$ ← From formula book.

$= \frac{2}{3}ml^2$

Conservation of angular momentum:

$2m \times 2\sqrt{gl} \times l\sqrt{2} = \left[\frac{2ml^2}{3} + 2m(l\sqrt{2})^2\right]\omega$ ← Diagonal of the square is $2l\sqrt{2}$. ← Remember to include the M.I. of the particle.

$4ml\sqrt{2gl} = \frac{14ml^2}{3}\omega$

$\omega = \frac{12\sqrt{2gl}}{14l} = \frac{6}{7}\sqrt{\frac{2g}{l}}$ ← Equivalent forms of this answer are acceptable.

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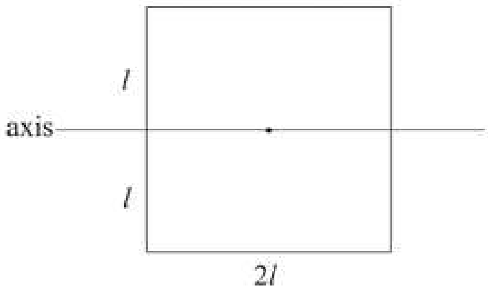
Exercise C, Question 9

Question:

A uniform square lamina of mass m and side $2l$ is rotating with angular speed $\sqrt{\frac{6g}{l}}$ about a fixed smooth horizontal axis through the centre of the lamina parallel to one side of the lamina. A particle of mass $2m$ is held at a height $12l$ above the level of the axis of rotation of the lamina. The particle is released from rest and hits the lamina at an instant when the lamina is horizontal. The particle adheres to the lamina at the mid-point of a side which is moving downwards at the instant of impact. Find the angular speed of the lamina immediately after the impact.

Solution:



M.I. of lamina $= \frac{1}{3}ml^2$ ← From the formula book.

Particle falling freely under gravity:

$s = 12l \quad u = 0$

$a = g \quad v^2 = u^2 + 2as$

$v^2 = 24gl$

For the impact:

Conservation of angular momentum:

$$2m\sqrt{24gl} \times l + \frac{1}{3}ml^2\left(\sqrt{\frac{6g}{l}}\right) = \left(2ml^2 + \frac{1}{3}ml^2\right)\omega$$

← The particle is a distance l from the axis.

$$2 \times 2\sqrt{6gl} + \frac{1}{3}\sqrt{6gl} = \frac{7}{3}l\omega$$

$$\frac{13\sqrt{6gl}}{3} = \frac{7}{3}l\omega$$

$$\omega = \frac{13}{7}\sqrt{\frac{6g}{l}}$$

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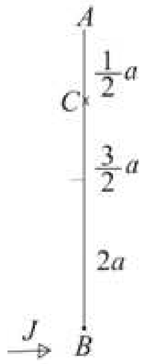
Exercise C, Question 10

Question:

A uniform rod AB of mass m and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through point C of the rod, where $AC = \frac{1}{2}a$. When the rod is hanging at rest with B vertically below A, the end B receives an impulse of magnitude J in a direction perpendicular to the axis of rotation. The impulse is sufficient to cause the rod to move in a complete circle. Show that the magnitude of the impulse is given by

$$J \geq \frac{m}{7}\sqrt{86ga}$$

Solution:



M.I. of rod about axis through $C = \frac{1}{3}m \times (2a)^2 + m\left(\frac{3}{2}a\right)^2$ ← From the formula book and using the parallel axes theorem.

$$= \frac{4ma^2}{3} + \frac{9ma^2}{4}$$

$$= \frac{43ma^2}{12}$$

For the impact:

$$J \times \frac{7a}{2} = \frac{43ma^2}{12}\omega$$ ← ω is the angular speed of the rod.

$$\omega = \frac{42J}{43ma}$$

Energy to top:

$$\frac{1}{2}\left(\frac{43ma^2}{12}\right) \times \left(\frac{42J}{43ma}\right)^2 - \frac{1}{2}\left(\frac{43ma^2}{12}\right)\omega_1^2$$ ← ω_1 is the angular speed of the rod when B is vertically above A.

$$= mg \times 3a$$

For complete circles $\omega_1 \geq 0$

$$\therefore \frac{1}{2}\left(\frac{43ma^2}{12}\right) \times \left(\frac{42J}{43ma}\right)^2 - 3mga \geq 0$$

$$\frac{1}{2} \times \frac{43}{12} \times \frac{42^2 J^2}{43^2 m} - 3mga \geq 0$$

$$J^2 \geq 3m^2 ga \times \frac{43 \times 24}{42^2}$$

$$J^2 \geq \frac{86}{49}m^2 ga$$

$$J \geq \frac{m}{7}\sqrt{86\,ga}$$ ← $J > 0$

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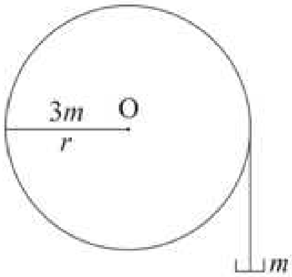
Exercise C, Question 11

Question:

A light inextensible string has one end attached to the rim of a pulley wheel of mass $3m$ and radius r. The string is wound several times around the wheel. A pan of mass m is attached to the other end of the string and hangs freely below the wheel. The system is held at rest. A particle of mass $5m$ is dropped from rest at a height $4r$ vertically above the pan. The particle adheres to the pan. The wheel is released from rest at the instant the particle hits the pan and begins to rotate about a fixed smooth horizontal axis through the centre of the wheel and perpendicular to the plane of the wheel. Assuming that the pulley wheel can be modelled as a uniform circular disc and the pan as a particle, find an expression for the angular speed of the wheel immediately after the impact.

Solution:



For the particle falling freely under gravity:

$s = 4r \quad v^2 = u^2 + 2as$

$a = g \quad v^2 = 2 \times 4rg = 8rg$

$u = 0$

M.I. of the wheel $= \frac{1}{2} \times 3mr^2$ ← From the formula book.

For the impact:

$5m \times \sqrt{8rg} \times r = \frac{3}{2}mr^2\omega + (5m + m)r\omega \times r$ ← Moment of the initial momentum of the particle = final angular momentum of the wheel + the moment of the final momentum of the pan and particle

$10\sqrt{2rg} = \frac{3}{2}r\omega + 6r\omega$

$10\sqrt{2rg} = \frac{15}{2}r\omega$

$\omega = \frac{20}{15}\sqrt{\frac{2g}{r}} = \frac{4}{3}\sqrt{\frac{2g}{r}}$

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Exercise C, Question 12

Question:

A uniform square lamina of mass m and side $2a$ is free to rotate about a fixed smooth horizontal axis which coincides with a side of the lamina. The lamina is hanging in equilibrium when it is hit at its centre of mass by a particle of mass $4m$ moving with speed v in a direction perpendicular to the plane of the lamina. The particle adheres to the lamina.

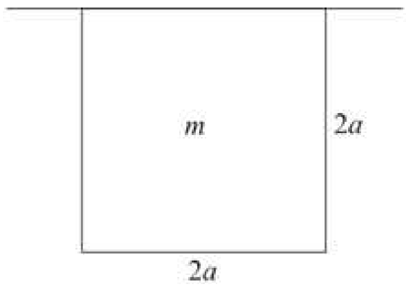
a Find the angular speed of the lamina immediately after the impact.

b Show that, for the lamina to move in a complete circle,

$$v \geq 22\sqrt{\left(\frac{5ga}{3}\right)}$$

Solution:

a



M.I. of lamina about axis along a side

$= \frac{4}{3}ma^2$ ← From the formula book.

For the impact:

$4mv \times a = \left(\frac{4}{3}ma^2 + 4ma^2\right)\omega$ ← Angular momentum is conserved.

$$4v = \frac{16a\omega}{3}$$

$$\omega = \frac{3v}{4a}$$

The angular speed is $\frac{3v}{4a}$.

b Energy:

$\frac{1}{2} \times \frac{16ma^2}{3} \times \left(\frac{3v}{4a}\right)^2 - \frac{1}{2} \times \frac{16ma^2}{3}\omega_1^2$ ← ω_1 is the angular speed when the lamina is vertical and above the axis.

$= 5mg \times 2a$

$$\frac{3v^2}{2} - \frac{8a^2}{3}\omega_1^2 = 10ag$$

For complete circles, $\omega_1^2 \geq 0$

$$\therefore \frac{3v^2}{2} - 10\,ag \geq 0$$

$$v^2 \geq \frac{2}{3} \times 10ag$$

$$v^2 \geq \frac{20ag}{3}$$

$$v \geq 2\sqrt{\frac{5ag}{3}}$$

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Exercise D, Question 1

Question:

A simple pendulum is performing small oscillations. Calculate the period of the pendulum when the length is

a 2.5 m,

b 0.8 m,

c 30 cm.

Solution:

a $T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{2.5}{9.8}}$

$T = 3.173\ldots$

$T = 3.2$ s (2 s.f.)

b $T = 2\pi\sqrt{\frac{0.8}{9.8}} = 1.795\ldots$

$T = 1.8$ s (2 s.f.)

c $T = 2\pi\sqrt{\frac{0.3}{9.8}} = 1.099\ldots$ ← Change cm to m.

$T = 1.1$ s (2 s.f.)

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Exercise D, Question 2

Question:

A simple pendulum is performing small oscillations. Calculate the length of the pendulum when the period is

a $\frac{1}{2}\pi$ s,

b $\frac{9}{16}\pi$ s,

c 0.8 s.

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$$

$$l = g\left(\frac{T}{2\pi}\right)^2$$

a $l = 9.8\left(\frac{\frac{1}{2}\pi}{2\pi}\right)^2 = \frac{9.8}{16} = 0.6125$

The length is 0.61 m (2 s.f.)

b $l = 9.8\left(\frac{\frac{9}{16}\pi}{2\pi}\right)^2 = 9.8 \times \left(\frac{9}{32}\right)^2$

$= 0.775\ldots$

The length is 0.78 m (2 s.f.)

c $l = 9.8\left(\frac{0.8}{2\pi}\right)^2 = 0.1588\ldots$

The length is 0.16 m (2 s.f.)

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Exercise D, Question 3

Question:

A simple pendulum has length a and period T. If the length is increased to $2a$, calculate the new period in terms of T.

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{a}{g}}$$

When length is $2a$:

$$T' = 2\pi\sqrt{\frac{2a}{g}}$$
$$= \left(2\pi\sqrt{\frac{a}{g}}\right) \times \sqrt{2}$$

$\therefore$ New period is $T\sqrt{2}$

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Exercise D, Question 4

Question:

A seconds pendulum takes one second to perform half an oscillation. Calculate the length of string required for this pendulum.

Solution:

Period = 2s

1 second for half an oscillation
$\therefore$ 2 seconds for a complete oscillation

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$2 = 2\pi\sqrt{\frac{l}{g}}$$

$$\left(\frac{1}{\pi}\right)^2 = \frac{l}{g}$$

$$l = \left(\frac{1}{\pi}\right)^2 \times 9.8 = 0.9929\ldots$$

The string must be 0.99 m long (2 s.f.)

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Exercise D, Question 5

Question:

A simple pendulum has length a and period T. Calculate, in terms of a, the length of a pendulum with period $\frac{1}{2}T$.

Solution:

$$T = 2\pi\sqrt{\frac{a}{g}}$$

$$\frac{1}{2}T = 2\pi\sqrt{\frac{l}{g}}$$

l is the length of the pendulum with period $\frac{1}{2}T$.

$$\therefore \frac{1}{2}\times 2\pi\sqrt{\frac{a}{g}} = 2\pi\sqrt{\frac{l}{g}}$$

$$\sqrt{\frac{a}{4g}} = \sqrt{\frac{l}{g}}$$

$$\therefore l = \frac{a}{4}$$

The length is $\frac{a}{4}$.

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Exercise D, Question 6

Question:

One end of a rope is tied to a branch of a tree. A girl is swinging on the other end of the rope. The period of oscillation is 2s. Assuming the girl and the rope can be modelled as a simple pendulum, calculate the length of the rope.
Calculate
a the period of small oscillations about the position of stable equilibrium,
b the length of the equivalent simple pendulum.

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$2 = 2\pi\sqrt{\frac{l}{g}}$$ ← The period is 2s.

$$\left(\frac{2}{2\pi}\right)^2 = \frac{l}{g}$$

$$l = \frac{9.8}{\pi^2} = 0.9929\ldots$$

The length of the rope is 0.99 m (2 s.f.)

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Exercise D, Question 7

Question:

A uniform rod, of mass m and length $6a$, is oscillating about a fixed smooth horizontal axis through one end of the rod.

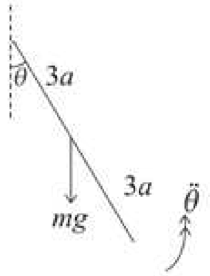
Calculate

a the period of small oscillations about the position of stable equilibrium,

b the length of the equivalent simple pendulum.

Solution:

a



M.I. of rod about axis through one end $= \frac{4}{3}m(3a)^2 = 12ma^2$ ← From the formula book

$mg \times 3a\sin\theta = -12ma^2\ddot{\theta}$ ← Use $L = I\ddot{\theta}$

For small θ, $\sin\theta \approx \theta$

$\therefore 12a^2\ddot{\theta} \approx -3ag\theta$

$\ddot{\theta} \approx -\frac{g}{4a}\theta$ ← The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4a}{g}}$

Period of small oscillations is $4\pi\sqrt{\frac{a}{g}}$

b The equivalent simple pendulum has length $4a$. ← Compare $T = 2\pi\sqrt{\frac{4a}{g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$.

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Exercise D, Question 8

Question:

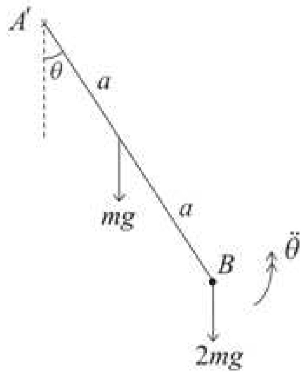
A uniform rod AB of mass m and length $2a$ with a particle of mass $2m$ attached at B, is oscillating about a fixed smooth perpendicular horizontal axis through A.
Calculate

a the period of small oscillations about the position of stable equilibrium,

b the length of the equivalent simple pendulum.

Solution:

a



M.I. of rod and particle about axis at $A = \frac{4}{3}ma^2 + 2m(2a)^2$

$= \frac{28}{3}ma^2$

$mga\sin\theta + 2mg \times 2a\sin\theta = -\frac{28}{3}ma^2\ddot{\theta}$ ← Use $L = I\ddot{\theta}$

$5g\sin\theta = -\frac{28}{3}a\ddot{\theta}$

For small θ, $\sin\theta \approx \theta$

$\therefore \ddot{\theta} \approx \frac{-15g}{28a}\theta$ ← The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{28a}{15g}}$

The period of small oscillations is $4\pi\sqrt{\frac{7a}{15g}}$

b The equivalent simple pendulum has length $\frac{28}{15}a$ ← Compare $T = 2\pi\sqrt{\frac{28a}{15g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$.

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Exercise D, Question 9

Question:

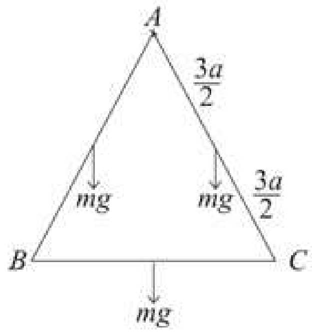
A triangular framework formed by joining three uniform rods, each of mass m and length $3a$, is oscillating about a fixed smooth horizontal axis through a vertex of the triangle perpendicular to the plane of the triangle.
Calculate

a the period of small oscillations about the position of stable equilibrium,

b the length of the equivalent simple pendulum.

Solution:

a



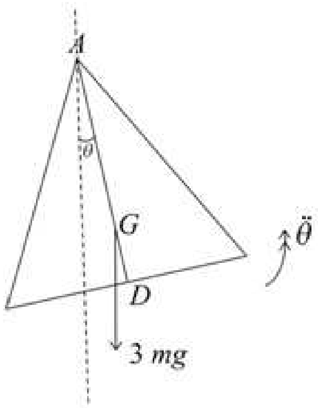
M.I. of framework about axis at A

$$= 2\times\frac{4}{3}m\left(\frac{3a}{2}\right)^2+\left[\frac{1}{3}m\left(\frac{3a}{2}\right)^2+m\left\{(3a)^2-\left(\frac{3a}{2}\right)^2\right\}\right]$$

Use the parallel axes theorem to obtain the M.I. of BC.

$$= 6ma^2+\frac{3ma^2}{4}+\frac{27ma^2}{4}$$

$$= \frac{27ma^2}{2}$$



Resultant force is $3mg$ at centre of mass of framework.

$AG = \frac{2}{3}AD = \frac{2}{3} \times \frac{3}{2}a\sqrt{3} = a\sqrt{3}$

$\therefore 3mg \times (a\sqrt{3})\sin\theta = \frac{-27}{2}ma^2\ddot{\theta}$

Use $L = I\ddot{\theta}$

$\ddot{\theta} = -\frac{2g}{9a}(\sqrt{3})\sin\theta$

For small oscillations $\sin\theta \approx \theta$

$\therefore \ddot{\theta} \approx -\frac{2g\sqrt{3}}{9a}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{9a}{2g\sqrt{3}}}$

The period of small oscillations is $6\pi\sqrt{\frac{a}{2g\sqrt{3}}}$.

b The equivalent simple pendulum has length $\frac{9a}{2\sqrt{3}}$ or $\frac{3\sqrt{3}}{2}a$

Compare $T = 2\pi\sqrt{\frac{9a}{2g\sqrt{3}}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

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Exercise D, Question 10

Question:

A uniform circular disc, of mass $2m$, radius a and centre O, with a particle of mass m attached at A, where $OA = \frac{1}{2}a$, is oscillating about a fixed smooth horizontal axis through O perpendicular to the disc.

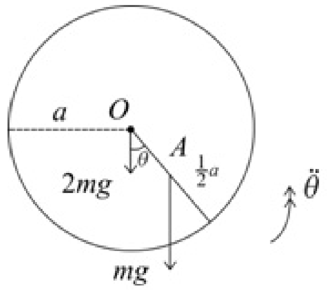
Calculate

a the period of small oscillations about the position of stable equilibrium,

b the length of the equivalent simple pendulum.

Solution:

a



M.I. of disc and particle about axis at $O = \frac{1}{2} \times 2ma^2 + m \times \left(\frac{1}{2}a\right)^2$

$$= \frac{5}{4}ma^2$$

$$mg \times \frac{1}{2}a \sin\theta = -\frac{5}{4}ma^2\ddot{\theta}$$ ← Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$\therefore g\theta \approx -\frac{5}{2}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{2g}{5a}\theta$$ ← The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5a}{2g}}$$

The period of small oscillations is $2\pi\sqrt{\frac{5a}{2g}}$.

b The equivalent simple pendulum has length $\frac{5a}{2}$.

Compare $T = 2\pi\sqrt{\frac{5a}{2g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

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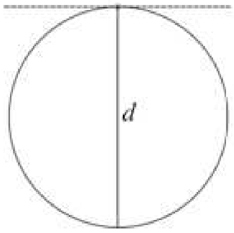
Exercise D, Question 11

Question:

A uniform circular hoop of mass m and diameter d is oscillating about a fixed smooth horizontal axis coinciding with a tangent to the hoop.
Calculate
a the period of small oscillations about the position of stable equilibrium,
b the length of the equivalent simple pendulum.

Solution:

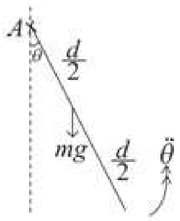
a



M.I. of hoop about a diameter $= \frac{1}{2}m\left(\frac{d}{2}\right)^2 = \frac{md^2}{8}$ ← From the formula book and perpendicular axes theorem.

M.I. of hoop about tangential axis $= \frac{md^2}{8} + m\left(\frac{d}{2}\right)^2 = \frac{3md^2}{8}$ ← By the parallel axes theorem.

side view



$mg\left(\frac{d}{2}\right)\sin\theta = -\frac{3md^2}{8}\ddot{\theta}$ ← Use $L = I\ddot{\theta}$

$g\sin\theta = -\frac{3d}{4}\ddot{\theta}$

For small θ $\sin\theta \approx \ddot{\theta}$ ← The motion is approximately simple harmonic.

$\therefore \ddot{\theta} \approx -\frac{4g}{3d}\theta$

$T = 2\pi\sqrt{\frac{3d}{4g}}$

The period of small oscillations is $\pi\sqrt{\frac{3d}{g}}$

b The equivalent simple pendulum has length $\frac{3d}{4}$. ← Compare $T = 2\pi\sqrt{\frac{3d}{4g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

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Exercise D, Question 12

Question:

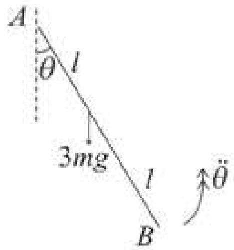
A uniform rod AB of mass $3m$ and length $2l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through A, perpendicular to the plane in which the rod rotates.

a Find the period of small oscillations of the rod about its position of equilibrium.

A particle of mass m is now attached to point B of the rod. The period of the oscillations is increased by x%.

b Find the value of x.

Solution:

a 

$$\text{M.I. of rod about axis through } A = \frac{4}{3} \times 3ml^2 = 4ml^2$$

From the formula book

$$3mgl\sin\theta = -4ml^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$\therefore 3g\theta \approx -4l\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{3g}{4l}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4l}{3g}} = 4\pi\sqrt{\frac{l}{3g}}$$

The period is $4\pi\sqrt{\frac{l}{3g}}$

b With a particle of mass m at B:

$$\text{M.I.} = 4ml^2 + m(2l)^2 = 8ml^2$$

$$3mgl\sin\theta + mg \times 2l\sin\theta = -8ml^2\ddot{\theta}$$

$$5g\theta \approx -8l\ddot{\theta}$$

As in **a** $\sin\theta \approx \theta$

$$\ddot{\theta} \approx -\frac{5g}{8l}\theta$$

$$\text{New period } = 2\pi\sqrt{\frac{8l}{5g}} = 4\pi\sqrt{\frac{2l}{5g}}$$

$$\therefore \%\text{increase} = \frac{\left(4\pi\sqrt{\frac{2l}{5g}} - 4\pi\sqrt{\frac{l}{3g}}\right)}{4\pi\sqrt{\frac{l}{3g}}} \times 100\%$$

4π and $\sqrt{\frac{l}{g}}$ will cancel.

$$= \frac{\sqrt{\frac{2}{5}} - \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}}} \times 100\%$$

$$= 9.544\ldots\%$$

$$\therefore x = 9.54 \text{ (3 s.f.)}$$

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Exercise D, Question 13

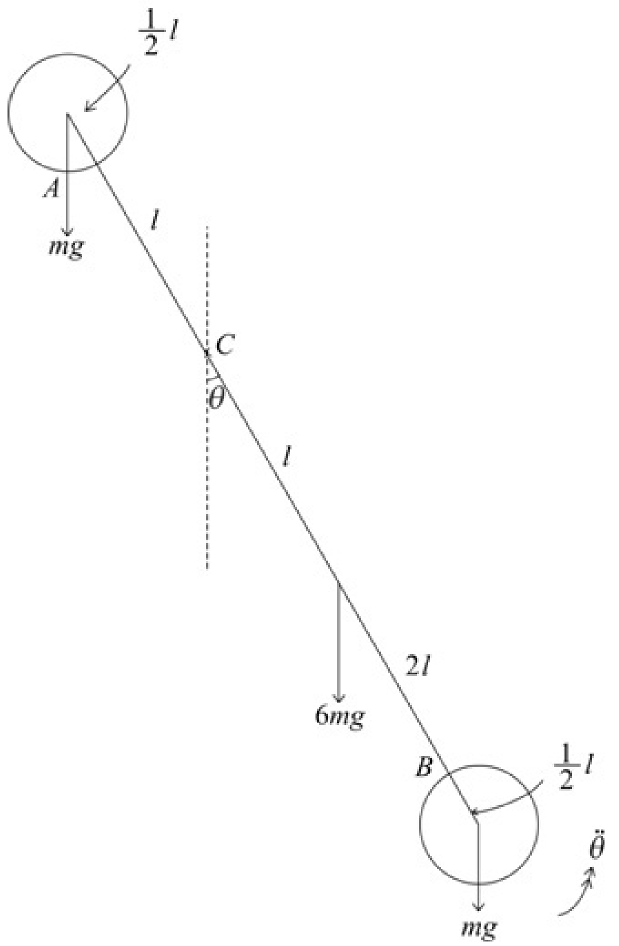
Question:

A uniform rod AB of mass $6m$ and length $4l$ has a uniform solid sphere attached to each end. Each sphere has mass m and radius $\frac{1}{2}l$ and the centres of both spheres lie on the same line as the rod. A fixed smooth horizontal axis passes through point C of the rod, where $AC = l$. The rod can rotate in a vertical plane which is perpendicular to this axis.

a Show that the moment of inertia of the system about the given axis is $\frac{287ml^2}{10}$.

b Find the period of small oscillations of the system about its position of stable equilibrium.

Solution:



a M.I. of rod about axis thro' $C = \frac{1}{3}(6m)(2l)^2 + 6ml^2$

$= 14ml^2$ ← Use the parallel axes theorem.

M.I. of spheres about axis thro' $C = \frac{2}{5}m\left(\frac{l}{2}\right)^2 + m\left(\frac{3}{2}l\right)^2 + \frac{2}{5}m\left(\frac{l}{2}\right)^2 + m\left(\frac{7l}{2}\right)^2$

$= 2 \times \frac{ml^2}{10} + \frac{58ml^2}{4}$

Use the parallel axes theorem again.

$\therefore$ Total M.I. of system

$= 14ml^2 + \frac{ml^2}{5} + \frac{29ml^2}{2}$

$= \frac{(140+2+145)}{10}ml^2$

$= \frac{287}{10}ml^2$

b $6mgl\sin\theta + mg\left(\frac{7l}{2}\right)\sin\theta - mg\left(\frac{3l}{2}\right)\sin\theta = -\frac{287ml^2}{10}\ddot{\theta}$

Use $L = I\ddot{\theta}$

$8g\sin\theta = -\frac{287}{10}l\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore 8g\theta \approx -\frac{287}{10}l\ddot{\theta}$

$\ddot{\theta} \approx -\frac{80g}{287l}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{287l}{80g}}$

The period of small oscillations is $2\pi\sqrt{\frac{287l}{80g}}$ $\left(\text{or } \frac{\pi}{2}\sqrt{\frac{287l}{5g}}\right)$

Any equivalent form is acceptable.

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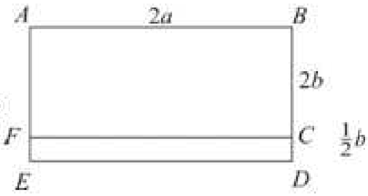
Exercise D, Question 14

Question:

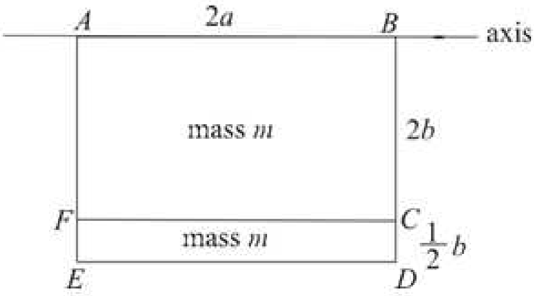
The diagram shows a rectangular sign outside a shop. The sign is composed of two portions, both of which are rectangular. Rectangle $ABCF$ has mass m, length $2a$ and width $2b$. Rectangle $FCDE$ has mass m, length $2a$ and width $\frac{1}{2}b$. The sign is free to rotate about a fixed smooth horizontal axis which coincides with side AB. The wind causes the sign to make small oscillations about its position of stable equilibrium.



Show that the period of these oscillations is given by $2\pi\sqrt{\frac{77b}{39g}}$.

[You may assume that both sections of the sign can be modelled as uniform rectangular laminae.]

Solution:



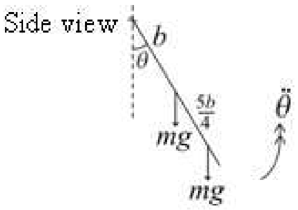
M.I. of sign about axis along AB

$$= \frac{4}{3}mb^2 + \frac{1}{3}m\left(\frac{1}{4}b\right)^2 + m\left(\frac{9}{4}b\right)^2$$

$$= mb^2\left(\frac{4}{3} + \frac{1}{48} + \frac{81}{16}\right)$$

$$= \frac{77mb^2}{12}$$

Use the parallel axes theorem for $CDEF$. Remember to move from the centre of mass.



$$mgb\sin\theta + mg \times \frac{9b}{4}\sin\theta = -\frac{77}{12}mb^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations, $\sin\theta \approx \theta$

$$g\theta + \frac{9g\theta}{4} \approx \frac{-77}{12}b\ddot{\theta}$$

$$\frac{13}{4}g\theta \approx -\frac{77}{12}b\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{39g}{77b}\theta$$

The motion is approximately simple harmonic.

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{77b}{39g}}$$

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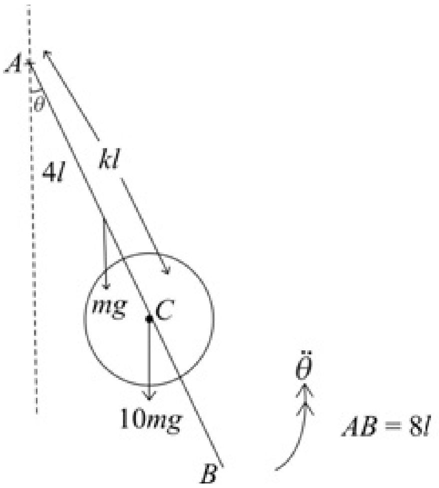
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 15

Question:

A thin uniform rod AB of mass m and length $8l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through end A. A uniform circular disc of radius $\frac{1}{2}l$ and mass $10m$ is clamped to the rod with its centre C on the rod and $AC = kl$. The plane of the disc coincides with the plane in which the rod can rotate and the axis is perpendicular to this plane.
Find the length of the equivalent simple pendulum.

Solution:



M.I. of rod and disc about axis at A

$$= \frac{4}{3}m(4l)^2 + \left[\frac{1}{2}\times 10m\left(\frac{1}{2}l\right)^2 + 10m(kl)^2\right]$$

Use the parallel axes theorem for the disc.

$$= \left(\frac{64}{3} + \frac{5}{4} + 10k^2\right)ml^2$$

$$= \left(\frac{271}{12} + 10k^2\right)ml^2$$

$$mg \times 4l\sin\theta + 10mg \times kl\sin\theta = -\left(\frac{271}{12} + 10k^2\right)ml^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$4g\theta + 10kg\theta \approx -\left(\frac{271}{12} + 10k^2\right)l\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{(4+10k)g\theta}{\left(\frac{271}{12} + 10k^2\right)l}$$

$$\ddot{\theta} \approx -\frac{24(2+5k)g}{(271+120k^2)l}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(271+120k^2)l}{24(2+5k)g}}$$

The equivalent simple pendulum has length $\dfrac{(271+120k^2)l}{24(2+5k)}$

Compare the expression for T with $T = 2\pi\sqrt{\dfrac{l}{g}}$

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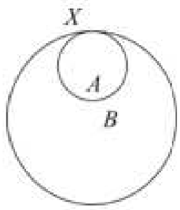
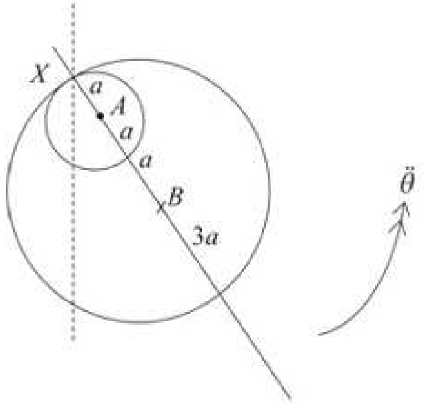
Exercise D, Question 16

Question:

An ear-ring of mass $8m$ is formed by cutting out a circle of radius a from a thin uniform circular disc of metal, radius $3a$, as shown in the diagram. The centre B of the larger circle, the centre A of the smaller circle and the point X on the circumference of both circles are collinear. The ear-ring is free to rotate about a fixed smooth horizontal axis through X perpendicular to the plane of the ear-ring. Show that the period of small oscillations of the ear-ring about its position of stable equilibrium is $4\pi\sqrt{\frac{15a}{13g}}$.



Solution:



Ratio of areas and masses	Cut-out circle	ear- ring	complete circle
	πa^2	$8\pi a^2$	$9\pi a^2$
	1	8	9

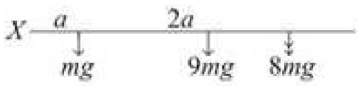
M.I. of complete disc about axis at $X = \frac{1}{2} \times 9m \times (3a)^2 + 9m \times (3a)^2$

$= \frac{27}{2} m \times (3a)^2 = \frac{243}{2} ma^2$

Use the parallel axes theorem.

M.I. of cut-out circle about axis at $X = \frac{1}{2} ma^2 + ma^2 = \frac{3ma^2}{2}$

$\therefore$ M.I. of ear-ring about axis at $X = \frac{243ma^2}{2} - \frac{3ma^2}{2} = 120ma^2$



You need to find the centre of mass of the ear-ring.

Ratio masses	1	8	9
Distance from X	a	$\bar{x}$	$3a$

$\therefore 8\bar{x} = 27a - a$

$\bar{x} = \frac{26a}{8}$

$8mg \times \frac{26a}{8} \sin\theta = -120ma^2\ddot{\theta}$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore 26g\theta \approx -120a\ddot{\theta}$

$\ddot{\theta} \approx \frac{-26g}{120a}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{120a}{26g}}$

$= 2\pi\sqrt{\frac{60a}{13g}}$

$= 4\pi\sqrt{\frac{15a}{13g}}$

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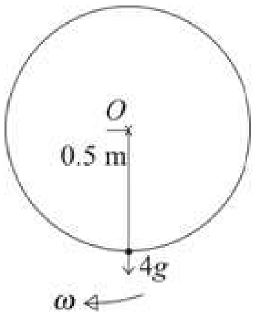
Exercise E, Question 1

Question:

A uniform circular disc of mass 20 kg and radius 0.5 m is free to rotate about a fixed smooth horizontal axis through its centre and perpendicular to its plane. A particle of mass 4 kg is attached to a point of the rim of the disc. Initially the disc is at rest in its position of unstable equilibrium. The disc is slightly disturbed. Find the angular speed of the disc at the moment when the particle is vertically below the axis.

Solution:



M.I. of disc + particle about axis through $O = \frac{1}{2} \times 20 \times (0.5)^2 + 4 \times (0.5)^2$

$= 3.5 \text{ kg m}^2$

Energy:

$\frac{1}{2} \times 3.5\omega^2 = 4g \times 1$

$\omega^2 = \frac{8g}{3.5}$

$\omega = 4.73\ldots$

The particle starts vertically above O and ends vertically below O.
K.E. $= \frac{1}{2} I\omega^2$

The angular speed is 4.7 rad s^{-1} (2 s.f.).

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Exercise E, Question 2

Question:

A uniform rod AB of mass m and length $2a$ is attached to a fixed smooth hinge at A. The rod is released from rest with AB horizontal. At time t the angle between the rod and the horizontal is θ.

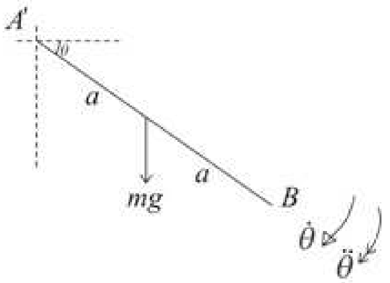
a Show that $2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$

b Find the magnitude of the component of the force exerted by the rod on the hinge parallel to the rod when $\theta = 45°$.

c Find the magnitude of the component of the force exerted by the rod on the hinge perpendicular to the rod when $\theta = 45°$.

Solution:

a



M.I. of rod about axis through $A = \frac{4}{3}ma^2$

Energy:

$$\frac{1}{2}I\dot{\theta}^2 = mga\sin\theta$$

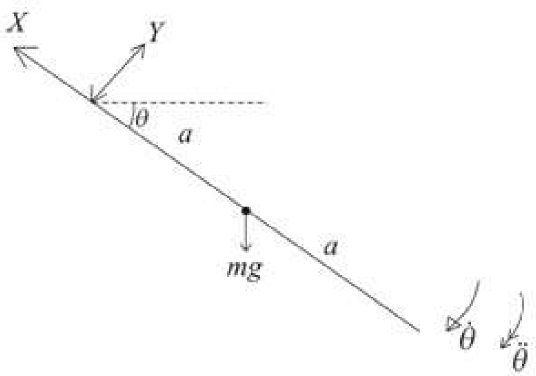
$$\frac{2}{3}ma^2\dot{\theta}^2 = mga\sin\theta$$

$$2a\dot{\theta}^2 = 3g\sin\theta$$

or $2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$

The rod starts from rest with AB horizontal.

b



Equation of motion along the radius:

$X - mg\sin\theta = ma\dot{\theta}^2$

$$X = mg\sin\theta + m\times\frac{3g}{2}\sin\theta$$

For the force on the axis of rotation you need to consider the motion of a particle of mass m at the centre of mass of the rod.

When $\theta = 45°$

$$X = mg\times\frac{1}{\sqrt{2}} + \frac{3mg}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{5mg}{2\sqrt{2}}$$

Use the result from **a**.

∴ The magnitude of the component of the force exerted by the rod on the hinge parallel to the rod is $\frac{5mg}{2\sqrt{2}}$

c Equation of motion perpendicular to the rod:

$mg\cos\theta - Y = ma\ddot{\theta}$

From **a**

$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$

$2 \times 2a\frac{d^2\theta}{dt^2} = 3g\cos\theta$

$\frac{d}{d\theta}(r\dot{\theta}^2) = 2r\ddot{\theta}$

$\therefore Y = mg\cos\theta - \frac{3mg}{4}\cos\theta$

$\theta = 45°\ Y = \frac{mg}{4} \times \frac{1}{\sqrt{2}} = \frac{mg}{4\sqrt{2}}$

$\therefore$ The magnitude of the component perpendicular to the rod is $\frac{mg}{4\sqrt{2}}$.

Solutionbank M5

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Exercise E, Question 3

Question:

A uniform circular disc of mass $4m$ and radius $2a$ hangs in equilibrium from a point A on its circumference. The disc is free to rotate about a fixed smooth horizontal axis which is tangential to the disc at A and lies in the plane of the disc. A particle P of mass m is moving horizontally towards the disc with speed V in a direction perpendicular to the plane of the disc. The particle strikes the disc at the point B where $AB = 3a$ and AB is perpendicular to the axis. The particle adheres to the disc.

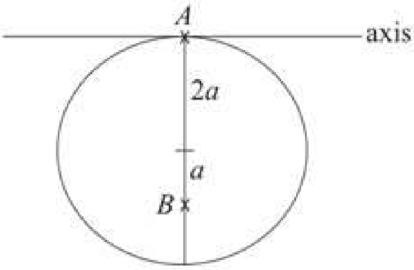
a Find the angular speed of the disc immediately after it has been struck by P.

The disc first comes to instantaneous rest when the angle between AB and the downward vertical at A is $60°$.

b Show that $V = \frac{1}{3}\sqrt{319ga}$.

Solution:

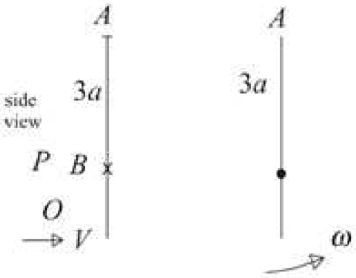
a



M.I. of disc about axis through A

$$= \frac{1}{4}4m(2a)^2 + 4m(2a)^2 = 20ma^2$$

Use the parallel axes theorem.



For the impact:

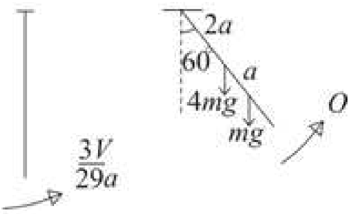
$$mV \times 3a = \left(20ma^2 + m \times (3a)^2\right)\omega$$

$$3V = 29a\omega$$

$$\omega = \frac{3V}{29a}$$

Angular momentum is conserved.

b



Energy to the highest point:

$$\frac{1}{2}\left(20ma^2 + 9ma^2\right)\left(\frac{3V}{29a}\right)^2 = 4mg \times 2a(1-\cos 60) + mg \times 3a(1-\cos 60)$$

$$\frac{1}{2} \times 29ma^2\left(\frac{3V}{29a}\right)^2 = 11mga \times \frac{1}{2}$$

$$\frac{9V^2}{29} = 11ga$$

$$V^2 = \frac{29 \times 11}{9}ga$$

$$V = \frac{1}{3}\sqrt{319\,ga}$$

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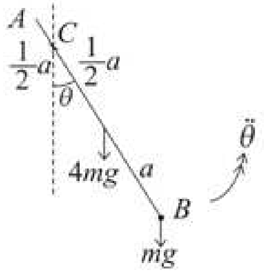
Exercise E, Question 4

Question:

A uniform rod AB of mass $4m$ and length $2a$ has a particle of mass m attached at B. The rod is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the rod and passing through point C of the rod where $AC = \frac{1}{2}a$. Find the period of small oscillations of the system about its position of stable equilibrium.

Solution:



M.I. of rod and particle about axis through C

$$= \left(\frac{1}{3}\times 4ma^2 + 4m\left(\frac{1}{2}a\right)^2\right) + m\left(\frac{3a}{2}\right)^2$$

$$= \frac{4}{3}ma^2 + ma^2 + \frac{9ma^2}{4}$$

$$= \frac{55ma^2}{12}$$

$$4mg \times \frac{1}{2}a\sin\theta + mg \times \frac{3a}{2}\sin\theta = -\frac{55}{12}ma^2\ddot{\theta}$$

$$\frac{7mga}{2}\sin\theta = -\frac{55}{12}ma^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$7g\theta \approx -\frac{55}{6}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{42g}{55a}\theta \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{55a}{42g}}$$

The motion is approximately simple harmonic.

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Exercise E, Question 5

Question:

A rough uniform rod, of mass m and length $6a$ is held on a rough horizontal table, perpendicular to the edge. A length $2a$ rests on the table and the remainder projects beyond the table.

a Find the moment of inertia of the rod about the edge of the table.

The rod is released from rest and rotates about the edge of the table. Assuming that the rod has not started to slip when it has turned through an angle θ,

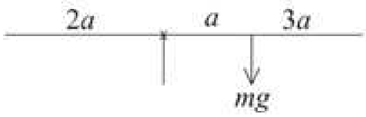
b find the angular acceleration of the rod,

c find the normal reaction of the table on the rod.

The coefficient of friction between the rod and the edge of the table is μ. The rod starts to slip when it makes an angle ϕ with the horizontal.

d Find tan ϕ in terms of μ.

Solution:

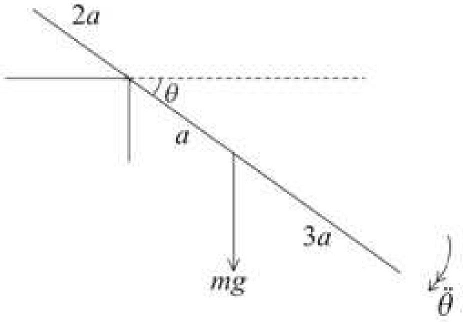


a M.I. of rod about edge of table

$= \frac{1}{3}m(3a)^2 + ma^2 = 4ma^2$

From the formula book and using the parallel axes theorem.

b

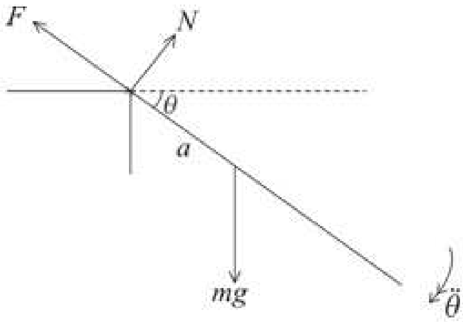


$mga\cos\theta = 4ma^2\ddot{\theta}$

$\ddot{\theta} = \frac{g}{4a}\cos\theta$

Use $L = I\ddot{\theta}$

c



Consider the motion of a particle of mass m at the mid-point of the rod.

Equation of motion perpendicular to the rod

$mg\cos\theta - N = ma\ddot{\theta}$

$mg\cos\theta - N = \frac{mg}{4}\cos\theta$

$N = \frac{3}{4}mg\cos\theta$

The normal reaction is $\frac{3}{4}mg\cos\theta$

d Equation of motion along the rod:

$F - mg\sin\theta = ma\dot{\theta}^2$

The component of the force parallel to the rod is needed so that you can find μ.

Energy:

$\frac{1}{2}\times 4ma^2\dot{\theta}^2 = mga\sin\theta$

Use conservation of energy to find $\dot{\theta}^2$

$a\dot{\theta}^2 = \frac{1}{2}g\sin\theta$

$\therefore F = \frac{1}{2}mg\sin\theta + mg\sin\theta$

$= \frac{3}{2}mg\sin\theta$

When $\theta = \phi, F = \mu N$

$\therefore \frac{3}{2}mg\sin\phi = \mu\times\frac{3}{4}mg\cos\phi$

The rod slips when $\theta = \phi$

$\tan\phi = \frac{1}{2}\mu$

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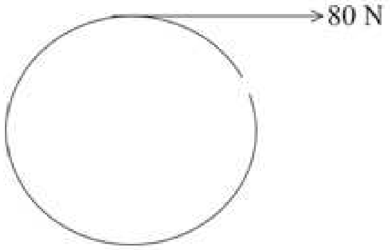
Exercise E, Question 6

Question:

A wheel has a rope of length 6 m wound round its axle. The rope is pulled with a constant force of 80 N. When the rope leaves the axle the wheel is rotating at 24 revolutions per minute. Calculate the moment of inertia of the wheel and its axle.

Solution:



$$\text{Work done by the force} = 80 \times 6$$
$$= 480 \text{ J}$$

K.E. gained by the wheel

$$= \frac{1}{2} I \omega^2$$

$\omega = 24$ revs. per minute

$$= \frac{24}{60} \times 2\pi = 0.8\pi \text{ rad s}^{-1}$$

$$\therefore \frac{1}{2} I \times (0.8\pi)^2 = 480$$

$$T = \frac{960}{0.64\pi^2} = 151.9\ldots$$

The moment of inertia is 152 kg m^2 (3 s.f.)

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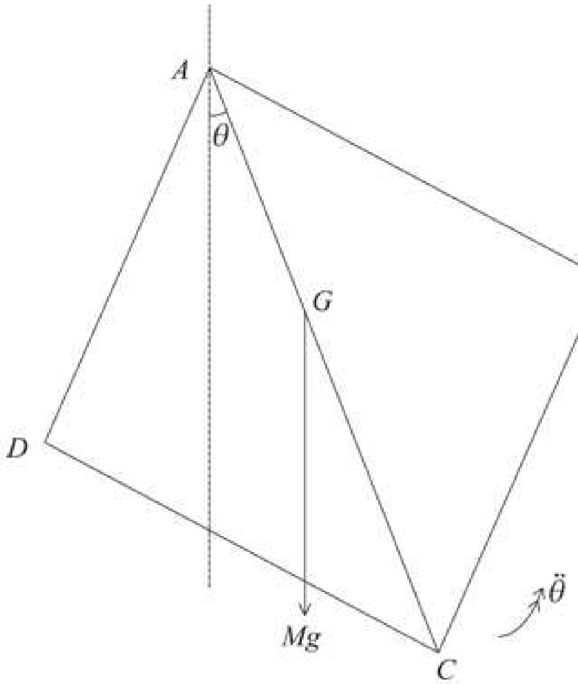
Exercise E, Question 7

Question:

A uniform square lamina $ABCD$ of mass M and side $2a$ is free to rotate about a fixed smooth horizontal axis through A. The axis is perpendicular to the plane of the lamina. The lamina is hanging at rest with C vertically below A. It is then disturbed from rest and performs small oscillations about its position of stable equilibrium. Find the period of these oscillations.

Solution:



M.I. of square about axis through A

$$= \frac{1}{3}M\left(a^2 + a^2\right) + M\left(a\sqrt{2}\right)^2$$

$$= \frac{2}{3}Ma^2 + 2Ma^2 = \frac{8Ma^2}{3}$$

From the formula book and using the parallel axes theorem $AG = a\sqrt{2}$

$$Mga\sqrt{2}\sin\theta = -\frac{8Ma^2}{3}\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations

$\sin\theta \approx \theta$

$$g\sqrt{2}\theta \approx -\frac{8}{3}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{3g\sqrt{2}}{8a}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{8a}{3g\sqrt{2}}} = 4\pi\sqrt{\frac{a\sqrt{2}}{3g}}$$

The period of small oscillations is $4\pi\sqrt{\frac{a\sqrt{2}}{3g}}$.

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Exercise E, Question 8

Question:

A uniform circular hoop of mass $4m$ and radius a is free to rotate in a vertical plane about a fixed smooth horizontal axis through point A of its circumference. The axis is perpendicular to the plane of the hoop and the hoop is initially hanging in equilibrium. A particle P of mass m is moving horizontally with speed V towards the hoop in the same plane as the hoop. The particle strikes the hoop at one end of its horizontal diameter and adheres to the hoop.

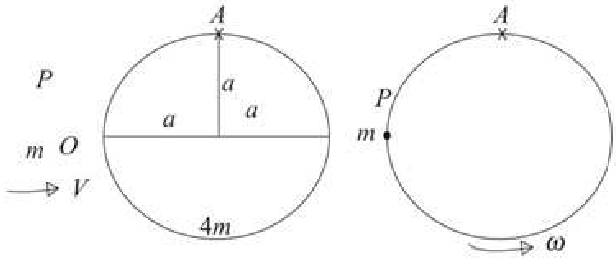
a Find the angular speed of the hoop immediately after P strikes it.

The line AB is a diameter of the hoop. The hoop first comes to instantaneous rest when AB is horizontal.

b Show that $V^2 = 80ga$

Solution:

a



M.I. of hoop about axis through $A = 4ma^2 + 4ma^2$
$= 8ma^2$

From the formula book and using the parallel axes theorem.

For the impact:

$$mVa = \left(8ma^2 + m\left(a\sqrt{2}\right)^2\right)\omega$$

Angular momentum is conserved.

$$V = (8a + 2a)\omega$$

$$\omega = \frac{V}{10a}$$

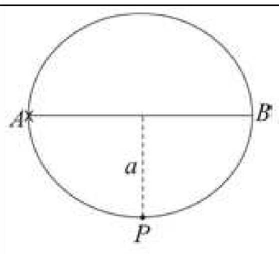
The angular speed is $\frac{V}{10a}$.

b Energy to highest point:

$$\frac{1}{2}\times 10ma^2\left(\frac{V}{10a}\right)^2 = 4mga$$

$$\frac{1}{2}\times\frac{V^2}{10} = 4ga$$

$$V^2 = 80\,ga$$



The vertical distance from A to P is the same at the start and the end.

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Exercise E, Question 9

Question:

A uniform rod AB of mass m and length $6l$ is free to rotate in a vertical plane perpendicular to a fixed smooth horizontal axis through point O of the rod, where $OA = l$. At time $t = 0$, the rod is at rest in its position of unstable equilibrium and is then slightly disturbed. At time t the rod has turned through an angle θ.

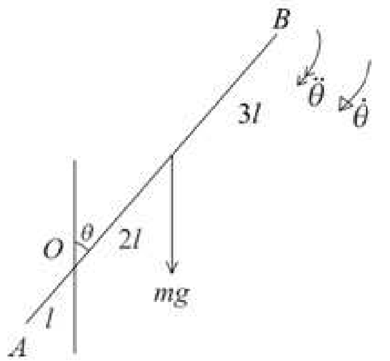
a Show that $7l\left(\frac{d\theta}{dt}\right)^2 = 4g(1-\cos\theta)$

b Find the magnitude of the angular acceleration of the rod at time t.

c Calculate the magnitude of the force exerted on the axis when the rod is horizontal.

Solution:

a



M.I. of rod about axis through $O = \frac{1}{3}m(3l)^2 + m(2l)^2$

$= 7ml^2$

From the formula book and using the parallel axes theorem.

Energy:

$\frac{1}{2}(7ml^2)\dot{\theta}^2 = mg \times 2l(1-\cos\theta)$

Increase in K.E. = loss of P.E.

$7l\dot{\theta}^2 = 4g(1-\cos\theta)$

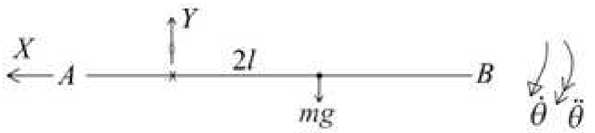
or $7l\left(\frac{d\theta}{dt}\right)^2 = 4g(1-\cos\theta)$

b $mg \times 2l\sin\theta = 7ml^2\ddot{\theta}$

$\ddot{\theta} = \frac{2g}{7l}\sin\theta$

Use $L = I\ddot{\theta}$
(or you can differentiate the result from **a** with respect to t).

c



Consider the motion of a particle of mass m at the centre of mass of the rod.

When the rod is horizontal $\theta = 90^\circ$

$$\therefore 7l\left(\frac{d\theta}{dt}\right)^2 = 4g$$

Use the result from **a**.

$$\text{and } \frac{d^2\theta}{dt^2} = \frac{2g}{7l}$$

Use the result from **b**.

Equation of motion for the particle parallel to AB:

$$X = m \times 2l\dot{\theta}^2$$

$$= 2m \times \frac{4g}{7} = \frac{8mg}{7}$$

Equation of motion perpendicular to AB:

$$mg - Y = m \times 2l\ddot{\theta}$$

$$Y = mg - 2ml \times \frac{2g}{7l} = \frac{3mg}{7}$$

$$\therefore \text{ Magnitude of the force on the axis} = \frac{mg}{7}\sqrt{(8^2 + 3^2)}$$

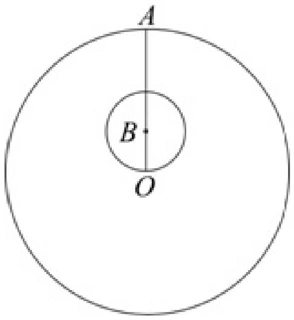
$$= \frac{mg\sqrt{73}}{7}$$

As the magnitude is required, the answer is the same for the force on the rod or the force on the axis.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 10

Question:



A uniform disc of mass $3m$ has centre O and radius $3a$. A disc with centre B and radius a is removed. The line $OB = a$ and, when produced, meets the circumference of the larger disc at A as shown in the diagram. The remaining lamina is free to rotate about a fixed smooth horizontal axis which coincides with the tangent to the disc at A.

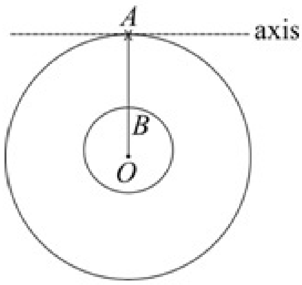
a Show that the moment of inertia of the remaining lamina about the given axis is $\frac{97ma^2}{3}$

The lamina is disturbed from rest and makes small oscillations about its position of stable equilibrium.

b Find the period of these oscillations.

Solution:

a



M.I. of complete disc about axis at A

$$= \frac{1}{4}(3m)(3a)^2 + 3m(3a)^2$$

$$= \frac{135ma^2}{4}$$

From the formula book, and using the parallel axes theorem.

M.I. of cut-out disc about axis at A

$$= \frac{1}{4}m_1a^2 + m_1(2a)^2$$

$$= \frac{17}{4}m_1a^2$$

m_1 is the mass of the smaller disc.

Area of complete disc $= 9\pi a^2$

Area of cut-out part $= \pi a^2$

$$\therefore\ m_1 = \frac{1}{9} \text{ of mass of complete disc}$$

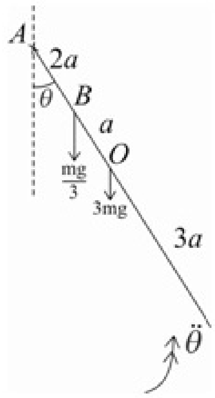
$$= \frac{1}{9} \times 3m = \frac{m}{3}$$

$\therefore$ M.I. of remaining lamina

$$= \frac{135}{4}ma^2 - \frac{17}{4} \times \frac{m}{3}a^2$$

$$= \frac{97}{3}ma^2$$

b Side view



Moment of the weight of the lamina about A

$$= 3mg \times 3a\sin\theta - \frac{mg}{3} \times 2a\sin\theta$$

$$= \frac{25mga}{3}\sin\theta$$

As you are using moments of forces you do not need to find the centre of mass of the lamina.

$$\therefore \frac{25}{3}mga\sin\theta = -\frac{97}{3}ma^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$25g\theta \approx -97a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{25g}{97a}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{97a}{25g}} = \frac{2\pi}{5}\sqrt{\frac{97a}{g}}$$

The period of small oscillations is $\frac{2\pi}{5}\sqrt{\frac{97a}{g}}$

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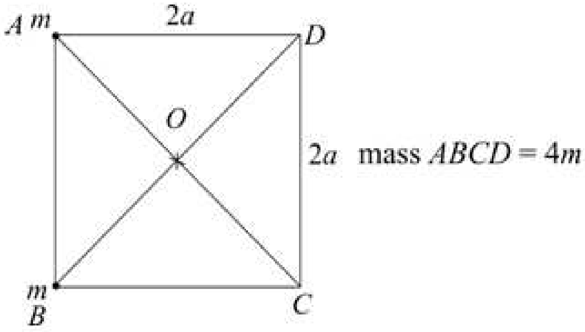
Exercise E, Question 11

Question:

A uniform square lamina $ABCD$ of mass $4m$ and side $2a$ is free to rotate in a vertical plane about a fixed smooth axis through its centre perpendicular to the plane of the lamina. Particles of mass m are attached to vertices A and B of the lamina. The system is released from rest with AB vertical. Find the angular speed of the system when AB is horizontal.

Solution:

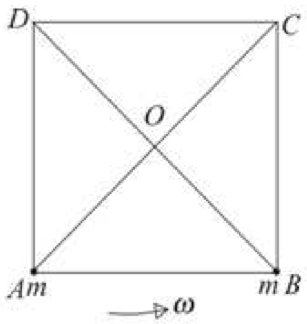


M.I. of lamina and particle about perpendicular axis through O

$$= \frac{1}{3} \times 4m\left(a^2 + a^2\right) + 2m \times 2a^2$$

$$= \frac{8ma^2}{3} + 4ma^2 = \frac{20ma^2}{3}$$

From formula book (lamina). $AO^2 = 2a^2$



Energy:

$$\frac{1}{2} \times \frac{20ma^2}{3} \omega^2 = mg \times 2a$$

Only the particle at A has experienced a change in P.E.

$$\frac{5}{3} a\omega^2 = g$$

$$\omega^2 = \frac{3g}{5a}$$

$$\omega = \sqrt{\frac{3g}{5a}}$$

The angular speed is $\sqrt{\frac{3g}{5a}}$.

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Exercise E, Question 12

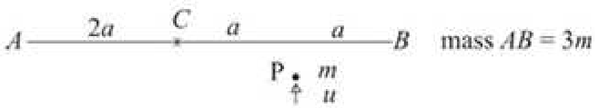
Question:

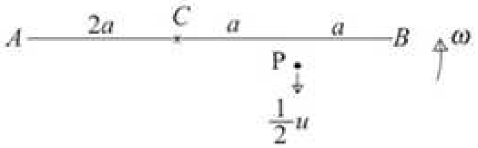
A uniform rod AB of mass $3m$ and length $4a$ lies at rest on a smooth horizontal plane. The rod is free to rotate about a fixed smooth vertical axis through its centre. A particle P of mass m is moving on the table with speed u in a direction perpendicular to the rod. The particle strikes the rod at a distance a from B and rebounds from the rod with its speed half of its speed before the collision.

a Find the angular speed of the rod after the collision.

b Show that there will not be a second collision between the rod and the particle.

Solution:





a For the impact:

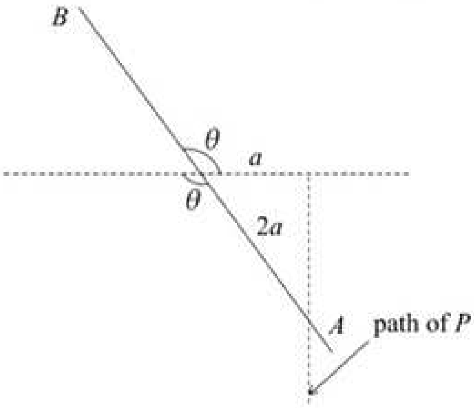
$$mua = \frac{1}{3} \times 3m(2a)^2\omega - \frac{1}{2}mua$$ ← Angular momentum is conserved.

$$4a^2\omega = \frac{3}{2}ua$$

$$\omega = \frac{3u}{8a}$$

The angular speed of the rod is $\frac{3u}{8a}$

b Time for the rod to turn through an angle $\theta = \frac{\theta}{\omega} = \frac{8a\theta}{3u}$



Distance travelled by P in this time $= \frac{1}{2}u \times \frac{8a\theta}{3u} = \frac{4a\theta}{3}$

For a second collision, there must be a $\theta, \frac{\pi}{2} < \theta < \pi$ such that

$$\sqrt{\left\{a^2 + \left(\frac{4a\theta}{3}\right)^2\right\}} \le 2a$$

$$1 + \frac{16\theta^2}{9} \le 4$$

$$\frac{16\theta^2}{9} \le 3$$

$$\theta^2 \le \frac{27}{16}$$

$$\therefore \theta \le 1.299$$

but $1.299 < \frac{\pi}{2}$ so there will not be another collision.

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Review Exercise 1
Exercise A, Question 1

Question:

At time t seconds a particle P has position vector $\mathbf{r}$ metres, relative to a fixed origin O.
The particle moves so that

$$\frac{d\mathbf{r}}{dt} - \mathbf{r} = 2e^{-t}\mathbf{i}.$$

When $t = 0$, $\mathbf{r} = -\mathbf{i} + \mathbf{j}$.
Find $\mathbf{r}$ in terms of t. ***E***

Solution:

$$\frac{d\mathbf{r}}{dt} - \mathbf{r} = 2e^{-t}\mathbf{i}$$

Integrating factor $= e^{\int -dt} = e^{-t}$

$$e^{-t}\frac{d\mathbf{r}}{dt} - \mathbf{r}e^{-t} = 2e^{-2t}\mathbf{i}$$

Multiply through by the integrating factor.

$$\frac{d}{dt}\left(\mathbf{r}e^{-t}\right) = 2e^{-2t}\mathbf{i}$$

$$\mathbf{r}e^{-t} = -e^{-2t}\mathbf{i} + \mathbf{c}$$

Integrate with respect to t. Don't forget the constant!

$$t = 0 \quad \mathbf{r} = -\mathbf{i} + \mathbf{j}$$

$$\Rightarrow -\mathbf{i} + \mathbf{j} = -\mathbf{i} + \mathbf{c}$$

$$\mathbf{c} = \mathbf{j}$$

Use the initial conditions given in the question to find $\mathbf{c}$.

$$\therefore \mathbf{r} = -e^{-t}\mathbf{i} + e^{t}\mathbf{j}$$

Multiply through by e^{t} to obtain $\mathbf{r}$.

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Review Exercise 1
Exercise A, Question 2

Question:

With respect to a fixed origin O, the position vector, $\mathbf{r}$ metres, of a particle P at time t seconds satisfies

$$\frac{d\mathbf{r}}{dt}+\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}.$$

Given that P is at O when $t=0$, find

a $\mathbf{r}$ in terms of t,

b a cartesian equation of the path of P. ***E***

Solution:

a $\frac{d\mathbf{r}}{dt}+\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}$

Integrating factor $= e^{\int 1\,dt}=e^{t}$

$e^{t}\frac{d\mathbf{r}}{dt}+e^{t}\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}\times e^{t}$ ← Multiply through by the integrating factor.

$\frac{d}{dt}(\mathbf{r}e^{t})=(\mathbf{i}-\mathbf{j})e^{-t}$

$\therefore \mathbf{r}e^{t}=-(\mathbf{i}-\mathbf{j})e^{-t}+\mathbf{c}$ ← Integrate with respect to t. Don't forget the constant!

$t=0, \mathbf{r}=\mathbf{0}\Rightarrow \mathbf{0}=-(\mathbf{i}-\mathbf{j})+\mathbf{c}$

$\mathbf{c}=\mathbf{i}-\mathbf{j}$ ← P is at O when $t=0$.

$\therefore \mathbf{r}=-(\mathbf{i}-\mathbf{j})e^{-2t}+(\mathbf{i}-\mathbf{j})e^{-t}$ ← Divide by e^{t} to obtain $\mathbf{r}$

b $\mathbf{r}=(-e^{-2t}+e^{-t})\mathbf{i}+(e^{-2t}-e^{-t})\mathbf{j}$

$x=-e^{-2t}+e^{-t}$ ← Using the $\mathbf{i}$ component.

$y=e^{-2t}-e^{-t}$ ← Using the $\mathbf{j}$ component.

$\therefore y=-x$ ← Eliminate t (by observation).

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Review Exercise 1
Exercise A, Question 3

Question:

At time t seconds the position vector of a particle P relative to a fixed origin O is $\mathbf{r}$ metres. The position vector satisfies the vector differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}.$$

At time $t = \frac{1}{2}\ln 3$, $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

a Find $\mathbf{r}$ in terms of t.

b Find the greatest value of the magnitude of the acceleration of P for $t \geq 0$. ***E***

Solution:

a $\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}$

Integrating factor $= e^{\int 2\,dt} = e^{2t}$

$\therefore e^{2t}\frac{d\mathbf{r}}{dt} + 2e^{2t}\mathbf{r} = \mathbf{0}$ — Multiply through by the integrating factor

$\frac{d}{dt}\left(e^{2t}\mathbf{r}\right) = \mathbf{0}$

$e^{2t}\mathbf{r} = \mathbf{A}$ — Integrate with respect to t.

$t = \frac{1}{2}\ln 3,\quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ — Use the initial conditions given in the question to find $\mathbf{A}$.

$e^{\ln 3}\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right) = \mathbf{A}$

$\mathbf{A} = 3\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right)$

$\therefore \mathbf{r} = 3e^{-2t}\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right)$

b $\dot{\mathbf{r}} = -6e^{-2t}\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right)$

$\ddot{\mathbf{r}} = 12e^{-2t}\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right)$

$|\ddot{\mathbf{r}}|_{max} = \left|12\left(\mathbf{i} - 2\mathbf{j} + \mathbf{k}\right)\right|$ — $|\ddot{\mathbf{r}}|$ will be maximum when $e^{-2t} = 1$

$= 12\sqrt{(1+4+1)}$

$= 12\sqrt{6}$

The greatest value of the magnitude of the acceleration is $12\sqrt{6}$ m s^{-2}.

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Review Exercise 1
Exercise A, Question 4

Question:

The position vector, $\mathbf{r}$ m, of a particle P is measured relative to a fixed origin O, and its velocity $\mathbf{v}$ m s^{-1} at time t seconds satisfies the differential equation

$$\frac{d\mathbf{v}}{dt} = -2\mathbf{v}.$$

When $t = 0$, P is at the point with position vector $(-2\mathbf{i}+\mathbf{j})$ m, and has velocity $(12\mathbf{i}+8\mathbf{j})$ m s^{-1}. Find

a an expression for $\mathbf{v}$ in terms of t,

b the position vector of P when $t = \ln 2$. ***E***

Solution:

a $\frac{d\mathbf{v}}{dt} = -2\mathbf{v}$

$\frac{d\mathbf{v}}{dt} + 2\mathbf{v} = \mathbf{0}$

Integrating factor $= e^{\int 2dt} = e^{2t}$

$e^{2t}\frac{d\mathbf{v}}{dt} + 2e^{2t}\mathbf{v} = \mathbf{0}$ ← Multiply through by the integrating factor

$\frac{d}{dt}\left(e^{2t}\mathbf{v}\right) = \mathbf{0}$

$e^{2t}\mathbf{v} = \mathbf{A}$ ← Integrate.

$t = 0, \mathbf{v} = 12\mathbf{i}+8\mathbf{j}$ ← Use the initial conditions given in the question to find $\mathbf{A}$.

$\Rightarrow 12\mathbf{i}+8\mathbf{j} = \mathbf{A}$

$\therefore \mathbf{v} = (12\mathbf{i}+8\mathbf{j})e^{-2t}$

b $\frac{d\mathbf{r}}{dt} = (12\mathbf{i}+8\mathbf{j})e^{-2t}$

$\mathbf{r} = -\frac{1}{2}(12\mathbf{i}+8\mathbf{j})e^{-2t} + \mathbf{B}$

$t = 0, \mathbf{r} = -2\mathbf{i}+\mathbf{j}$ ← Use the initial conditions given in the question to find $\mathbf{B}$.

$\Rightarrow -2\mathbf{i}+\mathbf{j} = -\frac{1}{2}(12\mathbf{i}+8\mathbf{j}) + \mathbf{B}$

$\mathbf{B} = 4\mathbf{i}+5\mathbf{j}$

$\therefore \mathbf{r} = -(6\mathbf{i}+4\mathbf{j})e^{-2t} + 4\mathbf{i}+5\mathbf{j}$

$t = \ln 2 \quad \mathbf{r} = -\frac{1}{4}(6\mathbf{i}+4\mathbf{j}) + 4\mathbf{i}+5\mathbf{j}$ ← $e^{-2\ln 2} = e^{\ln\left(\frac{1}{4}\right)} = \frac{1}{4}$

$\mathbf{r} = \frac{5}{2}\mathbf{i}+4\mathbf{j}$

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Review Exercise 1
Exercise A, Question 5

Question:

At time t seconds the position vector of a particle P, relative to a fixed origin O, is $\mathbf{r}$ metres, where $\mathbf{r}$ satisfies the differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = 3e^{-t}\mathbf{j}.$$

Given that $\mathbf{r} = 2\mathbf{i} - \mathbf{j}$ when $t = 0$, find $\mathbf{r}$ in terms of t. ***E***

Solution:

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = 3e^{-t}\mathbf{j}$$

Integrating factor $= e^{\int 2dt} = e^{2t}$

$$\therefore e^{2t}\frac{d\mathbf{r}}{dt} + 2\mathbf{r}e^{2t} = 3e^{t}\mathbf{j}$$

Multiply through by the integrating factor.

$$\frac{d}{dt}\left(\mathbf{r}e^{2t}\right) = 3e^{t}\mathbf{j}$$

$$\mathbf{r}e^{2t} = 3e^{t}\mathbf{j} + \mathbf{c}$$

Integrate with respect t. Don't forget the constant!

$$t = 0 \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j}$$

$$(2\mathbf{i} - \mathbf{j}) = 3\mathbf{j} + \mathbf{c}$$

Use the initial conditions given in the question.

$$\mathbf{c} = 2\mathbf{i} - 4\mathbf{j}$$

$$\therefore \mathbf{r} = 3e^{-t}\mathbf{j} + (2\mathbf{i} - 4\mathbf{j})e^{-2t}$$

Divide by e^{2t} to obtain $\mathbf{r}$.

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Review Exercise 1
Exercise A, Question 6

Question:

The position vector $\mathbf{r}$ metres of a particle P, relative to a fixed origin O, at time t seconds, satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 4\mathbf{r} = \mathbf{0}.$$

When $t = 0, \mathbf{r} = 3\mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 4\mathbf{j}$.

Find $\mathbf{r}$ in terms of t. **E**

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + 4\mathbf{r} = \mathbf{0}$$

Auxiliary equation : $m^2 + 4 = 0$

$$m = \pm 2\mathrm{i}$$

$$\therefore \mathbf{r} = \mathbf{A}\cos 2t + \mathbf{B}\sin 2t$$

$$t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \mathbf{A}$$

Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

$$\dot{\mathbf{r}} = -2\mathbf{A}\sin 2t + 2\mathbf{B}\cos 2t$$

$$t = 0, \dot{\mathbf{r}} = 2\mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow 2\mathbf{i} + 4\mathbf{j} = 2\mathbf{B}$$

$$\mathbf{B} = \mathbf{i} + 2\mathbf{j}$$

$$\therefore \mathbf{r} = 3\mathbf{i}\cos 2t + (\mathbf{i} + 2\mathbf{j})\sin 2t$$

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Review Exercise 1
Exercise A, Question 7

Question:

A particle P moves in a horizontal plane containing a fixed origin O. At time t, $\overrightarrow{OP} = \mathbf{r}$, where $\mathbf{r}$ satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0}.$$

At time $t = 0$ the particle is at the point with position vector $a\mathbf{j}$, and has velocity $\omega b\mathbf{i}$, where a, b and ω are constants.
Solve the differential equation to find $\mathbf{r}$ and hence find the cartesian equation of the path of the particle. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0}$$

Auxiliary equation : $m^2 + \omega^2 = 0$

$$m = \pm i\omega$$

$$\therefore \mathbf{r} = \mathbf{A}\cos\omega t + \mathbf{B}\sin\omega t$$

$$t = 0, \mathbf{r} = a\mathbf{j} \Rightarrow a\mathbf{j} = \mathbf{A}$$

$$\dot{\mathbf{r}} = -\mathbf{A}\omega\sin\omega t + \mathbf{B}\omega\cos\omega t$$

Use the initial conditions given in the question to find **A** and **B**.

$$t = 0, \dot{\mathbf{r}} = \omega b\mathbf{i}$$

$$\Rightarrow \omega b\mathbf{i} = \mathbf{B}\omega$$

$$\mathbf{B} = b\mathbf{i}$$

$$\therefore \mathbf{r} = b\sin\omega t\mathbf{i} + a\cos\omega t\mathbf{j}$$

hence $x = b\sin\omega t$

$y = a\cos\omega t$

$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = \sin^2\omega t + \cos^2\omega t$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

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Review Exercise 1
Exercise A, Question 8

Question:

At time t seconds, the position vector of a particle P is $\mathbf{r}$ metres, relative to a fixed origin. The particle moves in such a way that

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

At $t = 0$, P is moving with velocity $(8\mathbf{i} - 6\mathbf{j})$ m s^{-1}.

Find the speed of P when $t = \frac{1}{2}\ln 2$. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = \mathbf{0}$$

$$\frac{d\mathbf{v}}{dt} - 4\mathbf{v} = \mathbf{0}$$

You need to find the speed of P when $t = \frac{1}{2}\ln 2$ and speed $= |\text{velocity}|$. You do not need to find $\mathbf{r}$.

$$\text{Integrating factor} = e^{\int -4\,dt} = e^{-4t}$$

$$\therefore\ e^{-4t}\frac{d\mathbf{v}}{dt} - 4e^{-4t}\mathbf{v} = \mathbf{0}$$

Multiply through by the integrating factor.

$$\frac{d}{dt}\left(e^{-4t}\mathbf{v}\right) = \mathbf{0}$$

$$e^{-4t}\mathbf{v} = \mathbf{A}$$

Integrate with respect to t.

$$t = 0,\ \mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$$

$$\mathbf{A} = 8\mathbf{i} - 6\mathbf{j}$$

$$\therefore\ \mathbf{v} = (8\mathbf{i} - 6\mathbf{j})e^{4t}$$

$$t = \frac{1}{2}\ln 2,\ \mathbf{v} = 4(8\mathbf{i} - 6\mathbf{j})$$

$e^{2\ln 2} = e^{\ln 4} = 4$

$$\text{speed} = |\mathbf{v}| = 4\sqrt{(64+36)} = 40$$

The speed is 40 m s^{-1}.

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Review Exercise 1
Exercise A, Question 9

Question:

A particle P moves in the x-y plane and has position vector $\mathbf{r}$ metres at time t seconds. It is given that $\mathbf{r}$ satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}.$$

When $t = 0$, P is at the point with position vector $3\mathbf{i}$ metres and is moving with velocity $\mathbf{j}$ m s^{-1}.

a Find $\mathbf{r}$ in terms of t.

b Describe the path of P, giving its cartesian equation. ***E***

Solution:

a $\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}$

Auxiliary equation:

$m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0$ or $m = 2$

$\therefore \mathbf{r} = \mathbf{A}e^0 + \mathbf{B}e^{2t}$

$\mathbf{r} = \mathbf{A} + \mathbf{B}e^{2t}$

$t = 0, \mathbf{r} = 3\mathbf{i}$

$\Rightarrow \mathbf{A} + \mathbf{B} = 3\mathbf{i}$

$\dot{\mathbf{r}} = 2\mathbf{B}e^{2t}$

$t = 0, \dot{\mathbf{r}} = \mathbf{j} \Rightarrow 2\mathbf{B} = \mathbf{j}$

$\therefore \mathbf{B} = \frac{1}{2}\mathbf{j}$

$\mathbf{A} = 3\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\therefore \mathbf{r} = 3\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{j}e^{2t}$

or $\mathbf{r} = 3\mathbf{i} + \frac{1}{2}\mathbf{j}\left(e^{2t} - 1\right)$

> Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

b The particle moves in a straight line.
The equation of the line is $x = 3$.

> The $\mathbf{i}$ component is constant.

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Review Exercise 1
Exercise A, Question 10

Question:

At time t seconds, the position vector $\mathbf{r}$ metres of a particle P, relative to a fixed origin O, satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+3\mathbf{r}=\mathbf{0}.$$

At time $t=0$, P is at the point with position vector $2\mathbf{i}$ m and is moving with velocity $2\mathbf{j}$ m s^{-1}.

Find the position vector of P when $t=\ln 2$. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+3\mathbf{r}=\mathbf{0}$$

Auxiliary equation:

$m^2+4m+3=0$

$(m+3)(m+1)=0$

$\therefore m=-3$ or $m=-1$

$\mathbf{r}=\mathbf{A}e^{-t}+\mathbf{B}e^{-3t}$

$t=0, \mathbf{r}=2\mathbf{i} \Rightarrow 2\mathbf{i}=\mathbf{A}+\mathbf{B}$ ①

Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

$\dot{\mathbf{r}}=-\mathbf{A}e^{-t}-3\mathbf{B}e^{-3t}$

$t=0, \dot{\mathbf{r}}=2\mathbf{j} \Rightarrow 2\mathbf{j}=-\mathbf{A}-3\mathbf{B}$ ②

Solve ① and ② simultaneously.

$2\mathbf{i}+2\mathbf{j}=-2\mathbf{B}$

$\mathbf{B}=-(\mathbf{i}+\mathbf{j})$

$\therefore \mathbf{A}=2\mathbf{i}-\mathbf{B}=3\mathbf{i}+\mathbf{j}$

$\therefore \mathbf{r}=(3\mathbf{i}+\mathbf{j})e^{-t}-(\mathbf{i}+\mathbf{j})e^{-3t}$

$t=\ln 2 \Rightarrow e^{-t}=\frac{1}{2}$ and $e^{-3t}=\frac{1}{8}$

$e^{-\ln 2}=e^{\ln\left(\frac{1}{2}\right)}$

$\therefore$ When $t=\ln 2$

$\mathbf{r}=\frac{1}{2}(3\mathbf{i}+\mathbf{j})-\frac{1}{8}(\mathbf{i}+\mathbf{j})$

$\dot{\mathbf{r}}=\frac{11}{8}\mathbf{i}+\frac{3}{8}\mathbf{j}$

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Review Exercise 1
Exercise A, Question 11

Question:

A particle P of mass 2 kg moves in the x-y plane. At time t seconds its position vector is $\mathbf{r}$ metres. When $t = 0$, the position vector of P is $\mathbf{i}$ metres and the velocity of P is $(-\mathbf{i}+\mathbf{j})$ m s^{-1}.

The vector $\mathbf{r}$ satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+2\frac{d\mathbf{r}}{dt}+2\mathbf{r}=\mathbf{0}.$$

a Find $\mathbf{r}$ in terms of t.

b Show that the speed of P at time t is $e^{-t}\sqrt{2}$ m s^{-1}.

c Find, in terms of e, the loss of kinetic energy of P in the interval $t=0$ to $t=1$. ***E***

Solution:

a $\frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}$

Auxiliary equation:

$m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{(4-8)}}{2}$$

$$m = -1 \pm i$$

$\therefore \mathbf{r} = e^{-t}(\mathbf{A}\cos t + \mathbf{B}\sin t)$

$t = 0,\ \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{i} = \mathbf{A}$

Initial conditions are given in the question.

$\dot{\mathbf{r}} = -e^{-t}(\mathbf{A}\cos t + \mathbf{B}\sin t) + e^{-t}(-\mathbf{A}\sin t + \mathbf{B}\cos t)$

$t = 0,\ \dot{\mathbf{r}} = (-\mathbf{i} + \mathbf{j})$

$-\mathbf{i} + \mathbf{j} = -\mathbf{A} + \mathbf{B} \Rightarrow \mathbf{B} = \mathbf{j}$

$\therefore \mathbf{r} = e^{-t}(\cos t\mathbf{i} + \sin t\mathbf{j})$

b $\dot{\mathbf{r}} = -e^{-t}(\cos t\mathbf{i} + \sin t\mathbf{j}) + e^{-t}(-\sin t\mathbf{i} + \cos t\mathbf{j})$

$= (-e^{-t}\cos t - e^{-t}\sin t)\mathbf{i} + (-e^{-t}\sin t + e^{-t}\cos t)\mathbf{j}$

speed $= |\dot{\mathbf{r}}|$

$= e^{-t}\sqrt{\left((-\cos t - \sin t)^2 + (-\sin t + \cos t)^2\right)}$

$= e^{-t}\sqrt{\left(\cos^2 t + 2\cos t\sin t + \sin^2 t + \sin^2 t - 2\sin t\sin t + \cos^2 t\right)}$

$= e^{-t}\sqrt{2}\ \text{m s}^{-1}$

Using $\sin^2 t + \cos^2 t = 1$

c $t = 0$ speed $= \sqrt{2}$

$t = 1$ speed $= e^{-1}\sqrt{2} = \frac{\sqrt{2}}{e}$

$\therefore$ Loss of K.E. $= \frac{1}{2} \times 2 \times (\sqrt{2})^2 - \frac{1}{2} \times 2 \times \left(\frac{\sqrt{2}}{e}\right)^2$

$= 2 - \frac{2}{e^2}$ J

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Review Exercise 1
Exercise A, Question 12

Question:

A particle of mass 0.5 kg is at rest at the point with position vector $(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$ m. The particle is then acted upon by two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$. These are the only two forces acting on the particle.
Subsequently, the particle passes through the point with position vector $(4\mathbf{i}+5\mathbf{j}-5\mathbf{k})$ m with speed 12 m s^{-1}. Given that $\mathbf{F}_1=(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ N, find $\mathbf{F}_2$. ***E***

Solution:

$\mathbf{d}=(4\mathbf{i}+5\mathbf{j}-5\mathbf{k})-(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$

$\mathbf{d}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$

$\mathbf{F}\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=\frac{1}{2}\times\frac{1}{2}\times 12^2=36$

Work done $(\mathbf{F}\cdot\mathbf{d})$ = gain in K.E.

$\mathbf{F}=\lambda(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$\therefore \lambda(2\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=36$

$\lambda(4+4+1)=36$

$\lambda=4$

$\therefore \mathbf{F}=4(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

The particle starts at rest and $\therefore$ the resultant force acts along its path

But $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$

and $\mathbf{F}_1=\mathbf{i}+2\mathbf{j}-\mathbf{k}$

Given in the question.

$\therefore \mathbf{F}_2=8\mathbf{i}+8\mathbf{j}-4\mathbf{k}-\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\mathbf{F}_2=7\mathbf{i}+6\mathbf{j}-3\mathbf{k}$

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Review Exercise 1
Exercise A, Question 13

Question:

Two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are the only forces acting on a particle. $\mathbf{F}_1$ has magnitude 9 N and acts in the direction of $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$. $\mathbf{F}_2$ has magnitude 18 N and acts in the direction of $\mathbf{i}+8\mathbf{j}-4\mathbf{k}$.
Find the total work done by the two forces in moving the particle from the point with position vector $(\mathbf{i}+\mathbf{j}+\mathbf{k})$ m to the point with position vector $(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$ m. *E*

Solution:

$\mathbf{F}_1 = \lambda(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$

$|\mathbf{F}_1| = 9$

$|2\mathbf{i}+\mathbf{j}+2\mathbf{k}| = \sqrt{(4+1+4)} = 3$

$\therefore \lambda = 3$

$\therefore \mathbf{F}_1 = 6\mathbf{i}+3\mathbf{j}+6\mathbf{k}$

$\mathbf{F}_1$ is a scalar multiple of $(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$. Use this fact and the magnitudes of the vectors to find $\mathbf{F}_1$.

$\mathbf{F}_2 = \mu(\mathbf{i}+8\mathbf{j}-4\mathbf{k})$

$|\mathbf{F}_2| = 18$

$|\mathbf{i}+8\mathbf{j}-4\mathbf{k}| = \sqrt{(1+64+16)} = 9$

$\therefore \mu = 2$

$\therefore \mathbf{F}_2 = 2\mathbf{i}+16\mathbf{j}-8\mathbf{k}$

Now find $\mathbf{F}_2$ in the same way.

$\mathbf{F}_1+\mathbf{F}_2 = 8\mathbf{i}+19\mathbf{j}-2\mathbf{k}$

$\therefore$ work done $= (8\mathbf{i}+19\mathbf{j}-2\mathbf{k})\cdot[3\mathbf{i}+2\mathbf{j}-\mathbf{k}-(\mathbf{i}+\mathbf{j}+\mathbf{k})]$

$= (8\mathbf{i}+19\mathbf{j}-2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$

$= 16+19+4$

$= 39$

work done $= \mathbf{F}\cdot\mathbf{d}$

The work done is 39 J

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Review Exercise 1
Exercise A, Question 14

Question:

[*In this question* $\mathbf{i}$ *and* $\mathbf{j}$ *are horizontal unit vectors.*]
A small smooth ring of mass 0.5 kg moves along a smooth horizontal wire. The only forces acting on the ring are its weight, the normal reaction from the wire, and a constant force $(5\mathbf{i}+\mathbf{j}-3\mathbf{k})$ N. The ring is initially at rest at the point with position vector $(\mathbf{i}+\mathbf{j}+\mathbf{k})$ m, relative to a fixed origin.
Find the speed of the ring as it passes through the point with position vector $(3\mathbf{i}+\mathbf{k})$ m. ***E***

Solution:

$\mathbf{d} = (3\mathbf{i}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$= 2\mathbf{i}-\mathbf{j}$

work done $= (5\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j})$ ← The ring moves horizontally so its weight and the normal reaction from the wire do no work.
$= 10-1$
$= 9$

Gain of K.E. $= \frac{1}{2}\times 0.5v^2$ ← The ring starts from rest.

$\therefore \frac{1}{2}\times 0.5v^2 = 9$ ← Gain of K.E. = work done.

$v^2 = 36$
$v = 6$

The speed is 6 m s^{-1}.

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Review Exercise 1
Exercise A, Question 15

Question:

A smooth wire connects $A(0, 3, 0)$ to $B(2, 1, 4)$. The units of length on the x, y, and z axes are metres. A ring is threaded on the wire and a constant force is applied to the ring. The resultant of this force and the weight of the ring is $(\mathbf{i} - \mathbf{j} + \mathbf{k})$ N.

Find the increase in kinetic energy of the ring as it is moved from A to B. ***E***

Solution:

$$\mathbf{d} = (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) - (0\mathbf{i} + 3\mathbf{j} + 0\mathbf{k})$$

$$\mathbf{d} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\text{Work done} = \mathbf{F} \cdot \mathbf{d}$$

$$= (\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$$

$$= 2 + 2 + 4$$

$$= 8$$

Work done = increase in K.E.

$\therefore$ increase in kinetic energy is 8 J

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Review Exercise 1
Exercise A, Question 16

Question:

In this question $\mathbf{i}$ *and* $\mathbf{j}$ *are perpendicular horizontal unit vectors and* $\mathbf{k}$ *is a vertical unit vector.*
A bead of mass 0.125 kg moves along a smooth straight wire in the direction $\mathbf{i}+2\mathbf{j}$, from rest at the point A with position vector $(\mathbf{i}+3\mathbf{k})$ m, relative to a fixed origin O. The bead is acted on by three forces. These are a constant force $(-2\mathbf{i}+2\mathbf{j})$ N, the force exerted by the wire and its own weight. Given that the speed of the bead when it reaches the point B on the wire is 2 m s^{-1}, find the position vector of B relative to O.

E

Solution:

$\mathbf{d} = \lambda(\mathbf{i}+2\mathbf{j})$ — The direction of travel is $\mathbf{i}+2\mathbf{j}$

$$\text{work done} = \mathbf{F}\cdot\mathbf{d}$$
$$= (-2\mathbf{i}+2\mathbf{j})\cdot\lambda(\mathbf{i}+2\mathbf{j})$$

The bead moves horizontally so the force exerted by the wire and the weight of the bead do no work.

$$= -2\lambda+4\lambda$$
$$= 2\lambda$$

$$\text{K.E. gained} = \frac{1}{2}\times 0.125\times 2^2$$

The bead starts from rest.

work done = gain of K.E.

$$= \frac{1}{4}$$
$$\therefore 2\lambda = \frac{1}{4}$$
$$\lambda = \frac{1}{8}$$
$$\therefore \mathbf{d} = \frac{1}{8}(\mathbf{i}+2\mathbf{j})$$
$$\overrightarrow{OB} = \overrightarrow{OA}+\mathbf{d}$$
$$\overrightarrow{OB} = \mathbf{i}+3\mathbf{k}+\frac{1}{8}(\mathbf{i}+2\mathbf{j})$$
$$\overrightarrow{OB} = \frac{9}{8}\mathbf{i}+\frac{1}{4}\mathbf{j}+3\mathbf{k}$$

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Review Exercise 1
Exercise A, Question 17

Question:

A bead of mass 0.5 kg is threaded on a smooth straight wire. The forces acting on the bead are a constant force $(2\mathbf{i}+3\mathbf{j}+x\mathbf{k})$ N, its weight $(-4.9\mathbf{k})$ N, and the reaction on the bead from the wire.

a Explain why the reaction on the bead from the wire does no work as the bead moves along the wire.

The bead moves from the point A with position vector $(\mathbf{i}+\mathbf{j}-3\mathbf{k})$ m relative to a fixed origin O to the point B with position vector $(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$ m. The speed of the bead at A is 2 m s^{-1} and the speed of the bead at B is 4 m s^{-1}.

b Find the value of x. ***E***

Solution:

a The wire is smooth so the reaction is perpendicular to the wire and so does no work.

b
$$\mathbf{d} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (3\mathbf{i}-\mathbf{j}+2\mathbf{k})-(\mathbf{i}+\mathbf{j}-3\mathbf{k})$$
$$= 2\mathbf{i}-2\mathbf{j}+5\mathbf{k}$$
$$\mathbf{F} = 2\mathbf{i}+3\mathbf{j}+x\mathbf{k}+(-4.9\mathbf{k})$$
$$= 2\mathbf{i}+3\mathbf{j}+(x-4.9)\mathbf{k}$$
$$\mathbf{F}\cdot\mathbf{d} = (2\mathbf{i}+3\mathbf{j}+(x-4.9)\mathbf{k})\cdot(2\mathbf{i}-2\mathbf{j}+5\mathbf{k})$$

$\mathbf{F}\cdot\mathbf{d}$ = work done

$$= 4-6+5(x-4.9)$$
$$= 5x-26.5$$
$$\text{Gain of K.E.} = \frac{1}{2}\times 0.5\times 4^2 - \frac{1}{2}\times 0.5\times 2^2$$
$$= 3$$
$$\therefore 5x-26.5 = 3$$
$$5x = 29.5$$

work done = gain of K.E.

$$x = 5.9$$

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Review Exercise 1
Exercise A, Question 18

Question:

In this question $\mathbf{i}$ *and* $\mathbf{j}$ *are perpendicular unit vectors in a horizontal plane and* $\mathbf{k}$ *is a unit vector vertically upwards.*
A small smooth ring of mass 0.1 kg is threaded onto a smooth horizontal wire which is parallel to $(\mathbf{i}+2\mathbf{j})$. The only forces acting on the ring are its weight, the normal reaction from the wire and a constant force $(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$ N. The ring starts from rest at the point A on the wire, whose position vector relative to a fixed origin is $(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ m, and passes through the point B with speed 5 m s^{-1}. Find the position vector of B. ***E***

Solution:

$$\text{work done} = (\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot\overrightarrow{AB}$$

The ring moves horizontally so the reaction from the wire and the weight do no work.

$$\text{K.E. gained} = \frac{1}{2}\times 0.1\times 5^2 = 1.25$$

The ring starts from rest.

$$\overrightarrow{AB} = \lambda(\mathbf{i}+2\mathbf{j})$$

The wire is parallel to $(\mathbf{i}+2\mathbf{j})$.

$$\therefore (\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot\lambda(\mathbf{i}+2\mathbf{j}) = 1.25$$

work done = gain of K.E.

$$\lambda(1+4) = 1.25$$

$$\lambda = \frac{1.25}{5} = 0.25$$

$$\therefore \overrightarrow{AB} = \frac{1}{4}(\mathbf{i}+2\mathbf{j})$$

$$\overrightarrow{OB} = \overrightarrow{OA}+\overrightarrow{AB}$$

$$\overrightarrow{OB} = (2\mathbf{i}-2\mathbf{j}-3\mathbf{k})+\frac{1}{4}(\mathbf{i}+2\mathbf{j})$$

$$\overrightarrow{OB} = \frac{9}{4}\mathbf{i}-\frac{3}{2}\mathbf{j}-3\mathbf{k}$$

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Review Exercise 1
Exercise A, Question 19

Question:

A particle P of mass 4 kg is acted upon by the constant force $\mathbf{F} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ N. The force $\mathbf{F}$ is the resultant of all the forces acting on P, including its weight. Initially P is at rest at the point A with position vector $(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ m, relative to a fixed origin O. Under the action of $\mathbf{F}$, P moves to the point B with position vector $(7\mathbf{i} + 8\mathbf{j})$ m.

a Find the speed of P when it reaches B.

b Find the vector moment of $\mathbf{F}$ about the origin. ***E***

Solution:

a $\mathbf{d} = 7\mathbf{i} + 8\mathbf{j} - (\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$

work done $= (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$ ← work done $= \mathbf{F} \cdot \mathbf{d}$

$= 12 + 27 + 3$

$= 42$

K.E. gained $= \frac{1}{2} \times 4v^2$ ← The particle starts from rest.

$\therefore 2v^2 = 42$

$v^2 = 21$

$v = \sqrt{21}$

The speed of P at B is $\sqrt{21}$ m s^{-1} (or 4.6 m s^{-1})

b Vector moment $= \mathbf{r} \times \mathbf{F}$

$= (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 2 & 3 & -1 \end{vmatrix}$ ← The determinant method makes calculation of the vector product easier.

$= \mathbf{i}(1 - 9) - \mathbf{j}(-1 - 6) + \mathbf{k}(3 + 2)$

$= -8\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$

The vector moment is $(-8\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})$ Nm.

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Review Exercise 1
Exercise A, Question 20

Question:

Two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are the only forces acting on a particle P of mass 2 kg. The particle is initially at rest at the point A with position vector $(-2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$ m. Four seconds later, P is at the point B with position vector $(6\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})$ m.
Given that $\mathbf{F}_1 = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$ N, find

a $\mathbf{F}_2$,

b the work done on P as it moves from A to B. ***E***

Solution:

a $\mathbf{d} = (6\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}) - (-2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$

$= 8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ ← Use $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ to find the acceleration.

$8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} = \mathbf{0} + \frac{1}{2}\mathbf{a} \times 4^2$

$\mathbf{a} = \mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\therefore (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) + \mathbf{F}_2 = 2\left(\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\right)$ ← Using $\mathbf{F} = m\mathbf{a}$ where $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

$\therefore \mathbf{F}_2 = (-10\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$N

b Work done $= (\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathbf{d}$

$= 2\left(\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\right) \cdot (8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k})$

$= 2(8 + 2 + 18)$

$= 56$

The work done is 56 J.

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Review Exercise 1
Exercise A, Question 21

Question:

A particle P of mass 4 kg is constrained to move along a smooth straight horizontal wire. Relative to a fixed origin, the vector equation of the wire is $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})$ where $\mathbf{r}$ is measured in metres. The particle moves under the action of a constant force $(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ N, from the point A where $\lambda = 1$, to the point B where $\lambda = 3$. Given that the speed of P at B is 6 m s^{-1}, find the speed of P at A.

E

Solution:

$\mathbf{r}_A = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + (3\mathbf{i} - 4\mathbf{j})$

$\mathbf{r}_B = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + 3(3\mathbf{i} - 4\mathbf{j})$

$\therefore \mathbf{d} = 2(3\mathbf{i} - 4\mathbf{j}) = 6\mathbf{i} - 8\mathbf{j}$

work done $= (12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \cdot (6\mathbf{i} - 8\mathbf{j})$ ← work done $= \mathbf{F} \cdot \mathbf{d}$

$= 72 - 32$

$= 40$

Gain of K.E. $= \frac{1}{2} \times 4 \times 6^2 - \frac{1}{2} \times 4 \times v^2$ ← work done = gain of K.E.

$\therefore 72 - 2v^2 = 40$

$2v^2 = 32$

$v^2 = 16$

$v = 4$

The speed of P at A is 4 m s^{-1}.

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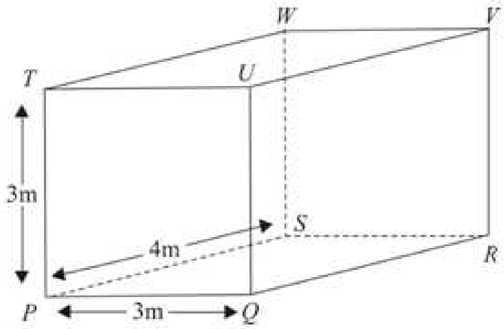
Review Exercise 1
Exercise A, Question 22

Question:



The diagram shows a box in the shape of a cuboid $PQRSTUVW$ where $\overrightarrow{PQ} = 3\mathbf{i}$ metres, $\overrightarrow{PS} = 4\mathbf{j}$ metres and $\overrightarrow{PT} = 3\mathbf{k}$ metres. A force $(4\mathbf{i} - 2\mathbf{j})$ N acts at Q, a force $(4\mathbf{i} + 2\mathbf{j})$ N acts at R, a force $(-2\mathbf{j} + \mathbf{k})$ N acts at T, and a force $(2\mathbf{j} + \mathbf{k})$ N acts at W. Given that these are the only forces acting on the box, find

a the resultant force acting on the box,

b the resultant vector moment about P of the four forces acting on the box.

When an additional force **F** acts on the box at a point X on the edge PS, the box is in equilibrium.

c Find **F**.

d Find the length of PX. ***E***

Solution:

> Moment = $\mathbf{r}\times\mathbf{F}$
> $\overrightarrow{PR} = \overrightarrow{PQ}+\overrightarrow{QR}$
> $\overrightarrow{PW} = \overrightarrow{PT}+\overrightarrow{TW}$

a $\mathbf{R} = (4\mathbf{i}-2\mathbf{j})+(4\mathbf{i}+2\mathbf{j})+(-2\mathbf{j}+\mathbf{k})+(2\mathbf{j}+\mathbf{k})$

$= 8\mathbf{i}+2\mathbf{k}$

The resultant is $= (8\mathbf{i}+2\mathbf{k})$ N

b Vector moment about P

$= 3\mathbf{i}\times(4\mathbf{i}-2\mathbf{j})+(3\mathbf{i}+4\mathbf{j})\times(4\mathbf{i}+2\mathbf{j})+3\mathbf{k}\times(-2\mathbf{j}+\mathbf{k})+(4\mathbf{j}+3\mathbf{k})\times(2\mathbf{j}+\mathbf{k})$

$= -6\mathbf{k}+(6\mathbf{k}-16\mathbf{k})+6\mathbf{i}+(4\mathbf{i}-6\mathbf{i})$

$= 4\mathbf{i}-16\mathbf{k}$

> Moment = $\mathbf{r}\times\mathbf{F}$
> $\overrightarrow{PR} = \overrightarrow{PQ}+\overrightarrow{QR}$
> $\overrightarrow{PW} = \overrightarrow{PT}+\overrightarrow{TW}$

The vector moment is $(4\mathbf{i}-16\mathbf{k})$ Nm

c For equilibrium, $\mathbf{F}+\mathbf{R}=0$

$\therefore \mathbf{F} = -8\mathbf{i}-2\mathbf{k}$

$\mathbf{E} = (-8\mathbf{i}-2\mathbf{k})$N

d For equilibrium, $\mathbf{r}\times\mathbf{F} = -4\mathbf{i}+16\mathbf{k}$

> $\mathbf{r} = \overrightarrow{PX}$ and moment of $\mathbf{F}$ + vector moment from part **b** must $=\mathbf{0}$ for equilibrium.

$\mathbf{r}\times(-8\mathbf{i}-2\mathbf{k}) = -4\mathbf{i}+16\mathbf{k}$

$\overrightarrow{PX} = \lambda\mathbf{j}$

> X is on PS

$\therefore \lambda\mathbf{j}\times(-8\mathbf{i}-2\mathbf{k}) = -(4\mathbf{i}-16\mathbf{k})$

$\lambda(8\mathbf{k}-2\mathbf{i}) = -(4\mathbf{i}-16\mathbf{k})$

$\lambda = 2$

$\therefore$ length $PX = 2$ m

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Review Exercise 1
Exercise A, Question 23

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, and a couple $\mathbf{G}$ act on a rigid body. The force $\mathbf{F}_1 = (3\mathbf{i} + 4\mathbf{j})$ N acts through the point with position vector $2\mathbf{i}$ m and the force $\mathbf{F}_2 = (2\mathbf{i} - \mathbf{j} + \mathbf{k})$ N acts through the point with position vector $(\mathbf{i} + \mathbf{j})$ m, relative to a fixed origin O. The forces and couple are equivalent to a single force $\mathbf{F}$ acting through O.

a Find the force $\mathbf{F}$.

b Find $\mathbf{G}$ and show that it has magnitude $3\sqrt{3}$ Nm. ***E***

Solution:

a $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

$$= (3\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

b Vector moment of $\mathbf{F}_1$ and $\mathbf{F}_2$ about O ← Vector moment $= \mathbf{r} \times \mathbf{F}$

$$= 2\mathbf{i} \times (3\mathbf{i} + 4\mathbf{j}) + (\mathbf{i} + \mathbf{j}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= 8\mathbf{k} + (\mathbf{i} - \mathbf{j}(1) + \mathbf{k}(-1-2))$$
$$= 8\mathbf{k} + \mathbf{i} - \mathbf{j} - 3\mathbf{k}$$
$$= \mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

The forces and the couple are equivalent to a single force $\mathbf{F}$ acting through O.

$$\therefore \mathbf{i} - \mathbf{j} + 5\mathbf{k} + \mathbf{G} = \mathbf{0}$$
$$\mathbf{G} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$
$$|\mathbf{G}| = \sqrt{(1+1+25)}$$
$$= \sqrt{27}$$
$$= 3\sqrt{3}$$

$\therefore \mathbf{G}$ is $(-\mathbf{i} + \mathbf{j} - 5\mathbf{k})$ and it has magnitude $3\sqrt{3}$ Nm.

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Review Exercise 1
Exercise A, Question 24

Question:

Two forces $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ N and $(3\mathbf{i}-\mathbf{k})$ N act through a point O of a rigid body, which is also acted upon by a couple of moment $(\mathbf{i}+\mathbf{j}+3\mathbf{k})$ Nm .

a Show that the couple and forces are equivalent to a single resultant force **F**.

b Find a vector equation for the line of action of **F** in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, where **a** and **b** are constant vectors and λ is a parameter. ***E***

Solution:

a $\mathbf{F}_1=(\mathbf{i}+2\mathbf{j}-\mathbf{k})\text{N}$

$$\mathbf{F}_2=(3\mathbf{i}-\mathbf{k})\text{N}$$
$$\mathbf{G}=(\mathbf{i}+\mathbf{j}+3\mathbf{k})\text{Nm}$$
$$(\mathbf{F}_1+\mathbf{F}_2)\cdot\mathbf{G}=(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})$$
$$=4+2-6$$
$$=0$$

$\therefore$ The forces and the couple are equivalent to a single resultant force.

b $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2=(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})\text{N}$

So **F** is parallel to the vector $(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

Let **F** pass through the point with position vector $\mathbf{r}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ relative to O

Then $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})=(\mathbf{i}+\mathbf{j}+3\mathbf{k})$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ 4 & 2 & -2\end{vmatrix}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

$\mathbf{r}\times\mathbf{F}$ is the vector moment of **F** about O and so must equal the given couple.

$$(-2y-2z)\mathbf{i}-(-2x-4z)\mathbf{j}+(2x-4y)\mathbf{k}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

$$-2y-2z=1 \quad ①$$
$$2x+4z=1 \quad ②$$
$$2x-4y=3 \quad ③$$

Equate coefficients of **i**, **j** and **k**.

$$③-②: -4y-4z=2$$
$$-2y-2z=1$$

This is the same as ①, so y can be any value

Let $y=0$

then $z=\frac{-1}{2}$

$x=\frac{3}{2}$

F passes through $\left(\frac{3}{2},0,\frac{-1}{2}\right)$

$\therefore$ An equation of the line of action is $\mathbf{r}=\left(\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{k}\right)+\lambda(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

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Review Exercise 1
Exercise A, Question 25

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a rigid body. $\mathbf{F}_1 = (21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k})\text{N}$ and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j} + r\mathbf{k})\text{N}$, where p, q and r are constants. $\mathbf{F}_1$ acts through the point A with position vector $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\text{m}$, relative to a fixed origin O. $\mathbf{F}_2$ acts through the point B with position vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})\text{m}$ relative to O.
The two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent to a single force $(25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})\text{N}$, acting through O, together with a couple $\mathbf{G}$.

a Find the values of p, q and r.

b Find the magnitude of $\mathbf{G}$. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})\text{N}$

$(21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k}) + (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})$

$(21 + p)\mathbf{i} + (q - 12)\mathbf{j} + (12 + r)\mathbf{k} = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})$

$\therefore p = 4 \quad q = -2 \quad r = 0$

Equating coefficients of **i**, **j** and **k**

b $\mathbf{G} = \sum \mathbf{r} \times \mathbf{F} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k}) + (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} - 2\mathbf{j})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 21 & -12 & 12 \end{vmatrix} = -12\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & -2 & 0 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

$\therefore \mathbf{G} = (-10\mathbf{i} - 11\mathbf{j})\ \text{Nm}$

$|\mathbf{G}| = \sqrt{(10^2 + 11^2)} = \sqrt{221}$

The magnitude of $\mathbf{G}$ is $\sqrt{221}\ \text{Nm}$.

Remember to complete the work by finding the magnitude of **G**.

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Review Exercise 1
Exercise A, Question 26

Question:

A system of forces consists of a force $(\mathbf{i}+2\mathbf{k})$ N acting at the point with position vector $(-\mathbf{i}+3\mathbf{j})$ m and a force $(-\mathbf{j}+\mathbf{k})$ N acting at the point with position vector $(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ m. This system is equivalent to a single force $\mathbf{F}$ N acting at the point with position vector $(\mathbf{j}+2\mathbf{k})$ m together with a couple $\mathbf{G}$ Nm.

a Find $\mathbf{F}$.
b Find $\mathbf{G}$.
c Give a reason why the system cannot be reduced to a single force without a couple.

E

Solution:

a $\mathbf{F}_1 = (\mathbf{i}+2\mathbf{k})$ N

$\mathbf{F}_2 = (-\mathbf{j}+\mathbf{k})$ N

$\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2 = (\mathbf{i}-\mathbf{j}+3\mathbf{k})$ N

b $\therefore \Sigma \mathbf{r}_i \times \mathbf{F}_i = (-\mathbf{i}+3\mathbf{j})\times(\mathbf{i}+2\mathbf{k})+(2\mathbf{i}+\mathbf{j}+\mathbf{k})\times(-\mathbf{j}+\mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 6\mathbf{i}+2\mathbf{j}-3\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 2\mathbf{i}-2\mathbf{j}-2\mathbf{k}$$

$\Sigma \mathbf{r} \times \mathbf{F} = 8\mathbf{i}-5\mathbf{k}$

Vector moment of resultant

$= (\mathbf{j}+2\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})$ ← $\mathbf{F}$ acts at the point with position vector $(\mathbf{j}+2\mathbf{k})$.

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = 5\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

$\therefore 5\mathbf{i}+2\mathbf{j}-\mathbf{k}+\mathbf{G} = 8\mathbf{i}-5\mathbf{k}$ ← Moment of resultant force + couple $= \Sigma \mathbf{r}_i \times \mathbf{F}_i$

$\Rightarrow \mathbf{G} = 3\mathbf{i}-2\mathbf{j}-4\mathbf{k}$

c $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(8\mathbf{i}-5\mathbf{j}) = 8+5$ ← The resultant force and the couple must be perpendicular if the system is to be reduced to a single force without a couple.

$= 13 \neq 0$

$\therefore$ The system cannot be reduced to a single force without a couple.

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Review Exercise 1
Exercise A, Question 27

Question:

The three forces $\mathbf{F}_1 = (q\mathbf{j} + r\mathbf{k})$N, $\mathbf{F}_2 = (p\mathbf{i} + r\mathbf{k})$N and $\mathbf{F}_3 = (p\mathbf{i} + q\mathbf{j})$N, where p, q and r are non-zero constants, act on a rigid body. $\mathbf{F}_1$ acts at the point with position vector $p\mathbf{i}$ m relative to a fixed origin O. $\mathbf{F}_2$ acts at the point with position vector $q\mathbf{j}$ m relative to O. $\mathbf{F}_3$ acts at the point with position vector $r\mathbf{k}$ m relative to O.

a Show that the three forces are equivalent to a single non-zero force acting at O.

b Find the magnitude of this single force. ***E***

Solution:

a $\Sigma\mathbf{F}_i = (q\mathbf{j} + r\mathbf{k}) + (p\mathbf{i} + r\mathbf{k}) + (p\mathbf{i} + q\mathbf{j})$

$= 2(p\mathbf{i} + q\mathbf{j} + r\mathbf{k})$ ← This is the resultant of the three forces.

Vector moment of system about 0

$= \Sigma\mathbf{r}_i \times \mathbf{F}_i$

$= p\mathbf{i} \times (q\mathbf{j} + r\mathbf{k}) + q\mathbf{j} \times (p\mathbf{i} + r\mathbf{k}) + r\mathbf{k} \times (p\mathbf{i} + q\mathbf{j})$

$= (pq\mathbf{k} - pr\mathbf{j}) + (-pq\mathbf{k} + qr\mathbf{i}) + (pr\mathbf{j} - qr\mathbf{i})$

$= \mathbf{0}$

No moment about O. ← Resultant must act through O.

So system is equivalent to force $2(p\mathbf{i} + q\mathbf{j} + r\mathbf{k})$N through O.

b Magnitude of resultant $= 2\sqrt{(p^2 + q^2 + r^2)}$N.

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Review Exercise 1
Exercise A, Question 28

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a rigid body, where $\mathbf{F}_1 = (2\mathbf{j} + 3\mathbf{k})\text{N}$ and $\mathbf{F}_2 = (\mathbf{i} + 4\mathbf{k})\text{N}$. The force $\mathbf{F}_1$ acts through the point with position vector $(\mathbf{i} + \mathbf{k})\text{m}$ relative to a fixed origin O. The force $\mathbf{F}_2$ acts through the point with position vector $(2\mathbf{j})\text{m}$. The two forces are equivalent to a single force $\mathbf{F}$.

a Find the magnitude of $\mathbf{F}$.

b Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, a vector equation of the line of action of $\mathbf{F}$.

E

Solution:

a $\mathbf{F} = \Sigma\mathbf{F}_i$

$= (2\mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + 4\mathbf{k})$

$= (\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})\text{N}$

$|\mathbf{F}| = \sqrt{(1 + 4 + 49)} = \sqrt{54}\text{ N}$

$= 3\sqrt{6}\text{ N}$

Remember to finish this part of the question by finding the magnitude of $\mathbf{F}$.

b Let $\mathbf{F}$ act through the point with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + 2\mathbf{k}$.

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$= (\mathbf{i} + \mathbf{k}) \times (2\mathbf{j} + 3\mathbf{k}) + 2\mathbf{j} \times (\mathbf{i} + 4\mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 2 & 7 \end{vmatrix} = (7y - 2z)\mathbf{i} - (7x - z)\mathbf{j} + (2x - y)\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 8\mathbf{i} - 2\mathbf{k}$$

$\therefore 7y - 2z = -2 + 8 = 6$ ①

$-7x + z = -3$ ②

$2x - y = 2 - 2 = 0$ ③

Equating coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

① + 2 ×②: $7y - 14x = 0$

$y = 2x$

Same as ③

$\therefore$ A suitable point is $(0, 0, -3)$

$\mathbf{F}$ is parallel to $(\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$\therefore$ An equation for the line of action of $\mathbf{F}$ is $\mathbf{r} = -3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

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Review Exercise 1
Exercise A, Question 29

Question:

Three forces, $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ act on a rigid body. $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\text{N}$, $\mathbf{F}_2 = (\mathbf{i} + \mathbf{j} - 4\mathbf{k})\text{N}$ and $\mathbf{F}_3 = (p\mathbf{i} + q\mathbf{j} + r\mathbf{k})\text{N}$, where p, q and r are constants. All three forces act through the point with position vector $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\text{m}$, relative to a fixed origin. The three forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ are equivalent to a single force $(5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})\text{N}$, acting at the origin, together with a couple $\mathbf{G}$.

a Find the values of p, q and r.

b Find $\mathbf{G}$. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ ← $\Sigma \mathbf{F}_i = \mathbf{F}$

$(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

$+ (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$(3 + p)\mathbf{i} + q\mathbf{j} + (r - 1)\mathbf{k} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$\Rightarrow p = 2, q = -4, r = 3$ ← Equate coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

b $\Sigma \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r} \times \mathbf{F}$

$= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ ← All three forces act through the same point. $\mathbf{F} = \Sigma \mathbf{F}_i$.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 5 & -4 & 2 \end{vmatrix} = 0\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$\therefore \mathbf{G} = (-\mathbf{j} - 2\mathbf{k})\text{Nm}$

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Review Exercise 1
Exercise A, Question 30

Question:

A force system consists of three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acting on a rigid body.

$\mathbf{F}_1 = (\mathbf{i} + 2\mathbf{j})$ N and acts at the point with position vector $(-\mathbf{i} + 4\mathbf{j})$ m.

$\mathbf{F}_2 = (-\mathbf{j} + \mathbf{k})$ N and acts at the point with position vector $(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ m.

$\mathbf{F}_3 = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$ N and acts at the point with position vector $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ m.

It is given that this system can be reduced to a single force $\mathbf{R}$.

a Find $\mathbf{R}$.

b Find a vector equation of the line of action of $\mathbf{R}$, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a parameter. ***E***

Solution:

a $\mathbf{R} = \Sigma \mathbf{F}_i$

$= (\mathbf{i} + 2\mathbf{j}) + (-\mathbf{j} + \mathbf{k}) + (3\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= (4\mathbf{i} + 2\mathbf{k})\,\text{N}$

b Let $\mathbf{R}$ act through a point with position vector $\mathbf{r} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$\mathbf{r} \times \mathbf{R} = \Sigma \mathbf{r}_i \times \mathbf{F}_i$

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (4\mathbf{i} + 2\mathbf{k})$

$= (-\mathbf{i} + 4\mathbf{j}) \times (\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (-\mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (3\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 4 & 0 & 2 \end{vmatrix} = 2y\mathbf{i} - (2x - 4z)\mathbf{j} - 4y\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -6\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

$\therefore 2y\mathbf{i} - (2x - 4z)\mathbf{j} - 4y\mathbf{k} = 3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$\therefore 2y = 3$ ①

$-2x + 4z = 3$ ②

$-4y = -6$ ③

$\therefore y = \frac{3}{2}$

Make $z = 0$ in ②, $x = -\frac{3}{2}$ $\therefore \left(-\frac{3}{2}, \frac{3}{2}, 0\right)$ lies on the line of action of $\mathbf{R}$.

$\mathbf{R} = (4\mathbf{i} + 2\mathbf{k})\,\text{N}$

$\therefore$ $\mathbf{R}$ is parallel to $2\mathbf{i} + \mathbf{k}$.

$\therefore$ An equation of the line of action of $\mathbf{R}$ is

$\mathbf{r} = \left(-\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \lambda(2\mathbf{i} + \mathbf{k})$.

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Review Exercise 1
Exercise A, Question 31

Question:

Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a rigid body. $\mathbf{F}_1 = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})\text{N}$ and acts at the point with position vector $(2\mathbf{i} - 3\mathbf{j})\text{m}$, $\mathbf{F}_2 = (-3\mathbf{j} + 2\mathbf{k})\text{N}$ and acts at the point with position vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})\text{m}$. The force $\mathbf{F}_3$ acts at the point with position vector $(2\mathbf{i} - \mathbf{k})\text{m}$.

Given that this set of forces is equivalent to a couple, find

a $\mathbf{F}_3$,

b the magnitude of the couple. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

$$\mathbf{F}_3 = -(12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) - (-3\mathbf{j} + 2\mathbf{k})$$
$$\mathbf{F}_3 = (-12\mathbf{i} + 7\mathbf{j} - 8\mathbf{k})\ \text{N}$$

b $\mathbf{G} = \Sigma \mathbf{r}_i \times \mathbf{F}_i$

$$\mathbf{r}_1 \times \mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 12 & -4 & 6 \end{vmatrix}$$
$$= -18\mathbf{i} - 12\mathbf{j} + 28\mathbf{k}$$

$$\mathbf{r}_2 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & -3 & 2 \end{vmatrix}$$
$$= 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{r}_3 \times \mathbf{F}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ -12 & 7 & -8 \end{vmatrix}$$
$$= 7\mathbf{i} + 28\mathbf{j} + 14\mathbf{k}$$

$$\therefore \mathbf{G} = (-6\mathbf{i} + 14\mathbf{j} + 39\mathbf{k})\ \text{Nm}$$
$$|\mathbf{G}| = \sqrt{(6^2 + 14^2 + 39^2)}$$
$$= 41.9\ \text{Nm} \quad (3\text{ s.f.})$$

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Review Exercise 1
Exercise A, Question 32

Question:

A spaceship is moving in a straight line in deep space and needs to reduce its speed from U to V. This is done by ejecting fuel from the front of the spaceship at a constant speed k relative to the spaceship. When the speed of the spaceship is v, its mass is m.

a Show that, while the spaceship is ejecting fuel, $\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m}{k}$.

The initial mass of the spaceship is M.

b Find, in terms of U, V, k and M, the amount of fuel which needs to be used to reduce the speed of the spaceship from U to V. ***E***

Solution:

a Conservation of momentum:

$$mv \approx (m+\delta m)(v+\delta v)+(-\delta m)(k+v+\delta v)$$
$$mv \approx mv + v\delta m + m\delta v + \delta m\delta v - k\delta m - v\delta m - \delta m\delta v$$
$$0 \approx m\delta v - k\delta m$$
$$k\delta m \approx m\delta v$$

In the limit, as $\delta t \to 0$

$$\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m}{k}$$

The fuel is ejected at a constant speed k relative to the space-ship. Its actual speed is therefore $(k+v+\delta v)$.

b
$$\int_M^{m_1} \frac{\mathrm{d}m}{m} = \int_U^V \frac{\mathrm{d}v}{k}$$

m_1 is the final mass of the space-ship

$$[\ln m]_M^{m_1} = \left[\frac{v}{k}\right]_U^V$$
$$\ln m_1 - \ln M = \frac{1}{k}(V-U)$$
$$\ln\left(\frac{m_1}{M}\right) = \frac{1}{k}(V-U)$$
$$m_1 = Me^{\frac{1}{k}(V-U)}$$

$$\text{Amount of fuel} = M - m_1 = M\left(1-e^{\frac{1}{k}(V-U)}\right)$$

The difference between the initial and final masses is the mass of the fuel ejected.

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Review Exercise 1
Exercise A, Question 33

Question:

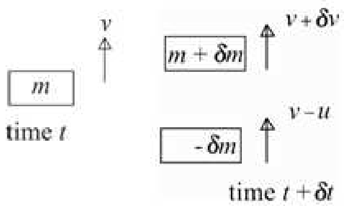
A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upward by ejecting burnt fuel vertically downwards at a constant speed u relative to the rocket. The initial mass of the rocket, including fuel, is M. At time t, before all the fuel has been used up, the mass of the rocket, including fuel, is $M(1-kt)$ and the speed of the rocket is v.

a Show that $\frac{dv}{dt} = \frac{ku}{1-kt} - g$.

b Hence find the speed of the rocket when $t = \frac{1}{3k}$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv = -mg\,\delta t$$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m+u\,\delta m-mv = -mg\,\delta t$$ ← Change in momentum = impulse

Let $\delta t \to 0$

$$m\frac{dv}{dt}+u\frac{dm}{dt} = -mg$$ ← $\frac{\delta m}{\delta t}\delta v \to 0$ when $\delta t \to 0$

$$m = M(1-kt) \Rightarrow \frac{dm}{dt} = -kM$$ ← Given in the question.

$$\therefore M(1-kt)\frac{dv}{dt}+u(-kM) = -M(1-kt)g$$

$$(1-kt)\frac{dv}{dt}-uk = -(1-kt)g$$

$$\therefore \frac{dv}{dt} = \frac{ku}{1-kt}-g$$

b $v = \int_0^{\frac{1}{3k}} \left(\frac{ku}{1-kt} - g\right) dt$ ← Rocket starts from rest.

$$= \left[-u\ln(1-kt) - gt\right]_0^{\frac{1}{3k}}$$

$$= -u\ln\left(1-\frac{1}{3}\right) - \frac{g}{3k}$$

The speed is $u\ln\left(\frac{3}{2}\right) - \frac{g}{3k}$

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Review Exercise 1
Exercise A, Question 34

Question:

A raindrop falls vertically under gravity through a cloud which is at rest. As it falls, water from the cloud condenses onto the drop in such a way that the mass of the drop increases at a constant rate of 0.02 g s^{-1}. At time t seconds, the speed of the drop is $v\text{ m s}^{-1}$, and when $t = 0$ the mass of the drop is 0.06 g. It is assumed that the only external force acting on the drop is gravity.

a Show that v satisfies the differential equation

$$(3+t)\frac{dv}{dt}+v=9.8(3+t).$$

Given that when $t = 0$, the raindrop is at rest,

b find the speed of the raindrop when its mass is twice its initial mass. ***E***

Solution:

a $(m+\delta m)(v+\delta v)-mv = mg\,\delta t$

$mv + v\delta m + m\delta v + \delta m\delta v - mv = mg\,\delta t$

Change in momentum = impulse

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$$v\frac{dm}{dt}+m\frac{dv}{dt} = mg$$

$$\frac{dm}{dt} = 0.02$$

The mass of the raindrop increases at a constant rate of 0.02 g s^{-1} (see question). *Note* It is OK to work in grams here as the mass units on each side of the differential equation cancel out. Had kg been used an extra factor of 10^{-2} would have been introduced on each side of the equation and this would have cancelled out.

$$\Rightarrow m = 0.02t + c$$

$$t = 0,\ m = 0.06$$

$$m = 0.02t + 0.06$$

$$\therefore 0.02v + (0.02t+0.06)\frac{dv}{dt} = (0.02t+0.06)g$$

$$v + (t+3)\frac{dv}{dt} = (t+3)g$$

$$\text{or } (3+t)\frac{dv}{dt} + v = 9.8(3+t)$$

b $\frac{dv}{dt} + \frac{v}{3+t} = 9.8$

Integrating factor $= e^{\int\frac{1}{3+t}dt}$

$= e^{\ln(3+t)} = (3+t)$

$$\therefore (3+t)\frac{dv}{dt}+v = 9.8(3+t)$$

$$\frac{d}{dt}[v(3+t)] = 9.8(3+t)$$

$$v(3+t) = 29.4t + 4.9t^2 + c$$

Integrating both sides of the equation.

Initial mass $= 0.06$ g

$\frac{dm}{dt} = 0.02\text{ g s}^{-1}$

$\therefore$ Mass doubled when $t = 3$

$t = 0,\ v = 0\ \therefore\ c = 0$

$t = 3,\ 6v = 29.4\times 3 + 4.9\times 9$

$v = 22.05$

$\therefore$ The speed is 22.1 m s^{-1} (3 s.f.)

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Review Exercise 1
Exercise A, Question 35

Question:

A rocket has total initial mass M. It propels itself by burning fuel and ejecting the burnt matter at a uniform rate with constant speed u relative to the rocket. The total mass of fuel in the rocket is initially $\frac{1}{2}M$, and the fuel is all burnt up after a time T. The rocket is launched from rest vertically upwards from the surface of the Earth. It may be assumed that the acceleration due to gravity remains constant throughout the flight of the rocket, and that air resistance is negligible. At time t, the speed of the rocket is v.

a Show that, while the fuel is being burnt,

$$(2T-t)\frac{dv}{dt} = u - g(2T-t).$$

b Hence find the speed of the rocket at the instant when all the fuel has been burnt. ***E***

Solution:

a $m = M - kt$

m = mass of rocket at time t. The burnt matter is ejected at a uniform rate.

$t = T,\ m = \frac{1}{2}M$

$\frac{1}{2}M = M - kT$

At time T all the fuel has been burnt.

$k = \frac{M}{2T}$

$\therefore\ m = M - \frac{M}{2T}t$

$(m + \delta m)(v + \delta v) + (-\delta m)(v - u) - mv = -mg\,\delta t$

Change of momentum = impulse

$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + u\,\delta m - mv = -mg\,\delta t$

$m\delta v + u\,\delta m + \delta m\delta v = -mg\,\delta t$

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$

$M\left(1 - \frac{t}{2T}\right)\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{M}{2T}u = -Mg\left(1 - \frac{t}{2T}\right)$

$m = M\left(1 - \frac{t}{2T}\right)$ from above $\Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = \frac{-M}{2T}$

$\therefore\ (2T - t)\frac{\mathrm{d}v}{\mathrm{d}t} = u - g(2T - t)$

Cancel M and multiply through by $2T$ to obtain the required result.

b $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{u}{(2T - t)} - g$

$v = \int_0^T \left(\frac{u}{2T - t} - g\right)\mathrm{d}t$

The fuel is all burnt when $t = T$. (see question) and the rocket starts from rest.

$v = \left[-u\ln(2T - t) - gt\right]_0^T$

$v = -u\ln T - gT - (-u\ln 2T)$

$= u\ln\frac{2T}{T} - gT$

$= u\ln 2 - gT$

The speed at the instant when all the fuel has been burnt is $u\ln 2 - gT$.

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Review Exercise 1
Exercise A, Question 36

Question:

A rocket initially has total mass M. It propels itself by its motor ejecting burnt fuel. When all of its fuel has been burned its mass is $kM, k < 1$. It is moving with speed U when its motor is started. The burnt fuel is ejected with constant speed c, relative to the rocket, in a direction opposite to that of the rocket's motion. Assuming that the only force acting on the rocket is that due to the motor, find the speed of the rocket when all of its fuel has been burned. ***E***

Solution:

Actual speed of fuel ejected $= v - c$

$$(m + \delta m)(v + \delta v) + (-\delta m)(v - c) = mv$$

Momentum is conserved.

$$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + c\,\delta m = mv$$

$$m\delta v + c\,\delta m + \delta m\delta v = 0$$

Let $\delta t \to 0$

$$m + c\frac{\mathrm{d}m}{\mathrm{d}v} = 0$$

$\frac{\delta m\delta v}{\delta v} = \delta m \to 0$ when $\delta t \to 0$

$$c\int\frac{\mathrm{d}m}{m} = -\int \mathrm{d}v$$

$$c\ln m = -v + A$$

A is the constant of integration.

$$t = 0, m = M, v = U$$

$$A = U + c\ln M$$

$$\therefore v + c\ln m = U + c\ln M$$

When all the fuel is burned, $m = kM$

$$\therefore v = -c\ln kM + U + c\ln M$$

$$= -c\ln\frac{kM}{M} + U$$

The speed is $U - c\ln k$.

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Review Exercise 1
Exercise A, Question 37

Question:

A rocket is launched vertically upwards from rest. Initially, the total mass of the rocket and its fuel is 1000 kg. The rocket burns fuel at a rate of 10 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after one minute. At time t seconds, $t \leq 60$, after the launch, the speed of the rocket is v m s^{-1}. Air resistance is assumed to be negligible.

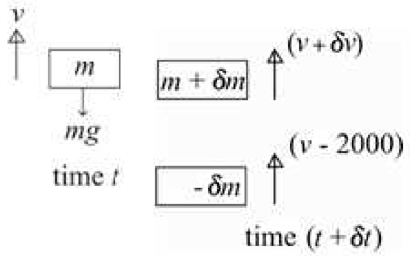
a Show that

$$-9.8(100-t) = (100-t)\frac{dv}{dt} - 2000.$$

b Find the speed of the rocket when burning stops. ***E***

Solution:

a



The fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket.

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv = -mg\delta t$$

$$mv+v\delta m+m\delta v+\delta m\delta v-v\delta m+2000\delta m-mv = -mg\delta t$$

Change of momentum = impulse

Let $\delta t \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$

$\frac{\delta m}{\delta t}\delta v \to 0$

But $m = 1000-10t$

$$\therefore \frac{\mathrm{d}m}{\mathrm{d}t} = -10$$

The initial total mass of the rocket and its fuel is 1000 kg and the fuel burns at 10 kg s^{-1}.

$$(1000-10t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times(-10) = -(1000-10t)g$$

$$(100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000 = -(100-t)\times 9.8$$

$$\text{or } -9.8(100-t) = (100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000$$

b $\frac{\mathrm{d}v}{\mathrm{d}t} = -9.8+\frac{2000}{(100-t)}$

$$v = -9.8t-2000\ln(100-t)+c$$

$t = 0, \quad v = 0$

$$\therefore \quad c = 2000\ln 100$$

$$v = 2000\ln 100-9.8t-2000\ln(100-t)$$

$$t = 60,\ v = 2000\ln 100-9.8\times 60-2000\ln 40$$

Burning stops after 1 minute (see question).

$$v = 2000\ln\frac{100}{40}-588$$

$$v = 2000\ln 2.5-588$$

$$v = 1244$$

When burning stops, the speed of the rocket is 1200 m s^{-1} (2 s.f.)

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Review Exercise 1
Exercise A, Question 38

Question:

A spherical raindrop falls under gravity through a stationary cloud. Initially the drop is at rest and its radius is a. As it falls, water from the cloud condenses on the drop in such a way that the radius of the drop increases at a constant rate k.
At time t, the speed of the drop is v.

a Show that

$$(a+kt)\frac{\mathrm{d}v}{\mathrm{d}t}+3kv=g(a+kt).$$

b Hence show that, when the drop has doubled its radius, it speed is $\frac{15ga}{32k}$. ***E***

Solution:

a $r = a + kt$ ← The initial radius is a and the radius increases at a constant rate k.

$(m+\delta m)(v+\delta v) - mv = mg\,\delta t$ ← Change of momentum = impulse.

$m = \frac{4}{3}\pi r^3 \rho$ ← The raindrop is spherical.

$\Rightarrow \frac{dm}{dt} = \frac{4}{3}\pi\rho \times 3r^2 \frac{dr}{dt}$

$= 4\pi\rho r^2 k$ ← $\frac{dr}{dt} = k$ (see question).

$\therefore mv + v\delta m + m\delta v + \delta m\delta v - mv = mg\,\delta t$

Let $\delta t \to 0$

$v\frac{dm}{dt} + m\frac{dv}{dt} = mg$ ← $\frac{\delta m}{\delta t}\delta v \to 0$

$v \times 4\pi\rho r^2 k + \frac{4}{3}\pi r^3 \rho \frac{dv}{dt} = \frac{4}{3}\pi r^3 \rho g$ ← Substitute for m and $\frac{dm}{dt}$.

$3vk + r\frac{dv}{dt} = rg$

$(a+kt)\frac{dv}{dt} + 3vk = g(a+kt)$ ← Substitute for r.

b $\frac{dv}{dt} + \frac{3vk}{(a+kt)} = g$

Integrating factor $= e^{3k\int \frac{dt}{(a+kt)}}$

$= e^{3\ln(a+kt)} = (a+kt)^3$

$(a+kt)^3\frac{dv}{dt} + 3vk(a+kt)^2 = g(a+kt)^3$

$\frac{d}{dt}\left[v(a+kt)^3\right] = g(a+kt)^3$

$v(a+kt)^3 = \frac{g}{4k}(a+kt)^4 + c$

$t = 0, v = 0 \Rightarrow 0 = \frac{ga^4}{4k} + c$

$\therefore v(a+kt)^3 = \frac{g}{4k}(a+kt)^4 - \frac{ga^4}{4k}$

radius doubled $\Rightarrow kt = a$

$\therefore v(2a)^3 = \frac{g}{4k}(2a)^4 - \frac{ga^4}{4k}$

$8a^3 v = \frac{4ga^4}{k} - \frac{ga^4}{4k}$

$8v = \frac{15ga}{4k}$

The speed is $\frac{15ga}{32k}$

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Review Exercise 1
Exercise A, Question 39

Question:

A hailstone falls under gravity in still air and as it falls its mass increases. Its initial mass is m_0. The rate of increase of its mass is proportional to its speed v.

a Show that, when the hailstone has fallen a distance x, it mass m is given by

$m = m_0(1+\lambda x)$, where λ is a constant.

Assuming that there is no air resistance,

b Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(v^2) + \frac{2\lambda}{1+\lambda x}(v^2) = 2g.$$

Given that $v = 0$ when $x = 0$,

c find an expression for v^2 in terms of x, λ and g. ***E***

Solution:

a $\frac{dm}{dt} = kv = k\frac{dx}{dt}$

The rate of increase of the hailstone's mass is proportional to its speed.

$$\int dm = k\int dx$$

$$m = kx + c$$

$$x = 0,\ m = m_0 \Rightarrow c = m_0$$

$$\therefore m = kx + m_0$$

$$m = m_0\left(1 + \frac{k}{m_0}x\right)$$

writing $\lambda = \frac{k}{m_0}$ gives

$$m = m_0(1 + \lambda x)$$

b $(m + \delta m)(v + \delta v) - mv = mg\,\delta t$

Change in momentum = impulse

$$mv + v\delta m + m\delta v + \delta m\delta v - mv = mg\,\delta t$$

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$$v\frac{dm}{dt} + m\frac{dv}{dt} = mg$$

$$v\frac{dm}{dt} + mv\frac{dv}{dt} = mg$$

The required result contains $\frac{d}{dx}(v^2)$, not $\frac{dv}{dt}$

But $m = m_0(1 + \lambda x)$

$$\Rightarrow \frac{dm}{dt} = m_0\lambda\frac{dx}{dt} = m_0\lambda v$$

$$\therefore vm_0\lambda v + mv\frac{dv}{dx} = mg$$

$$v^2\lambda\frac{m}{(1+\lambda x)} + \frac{1}{2}m\frac{d(v^2)}{dx} = mg$$

From **a** $m_0 = \frac{m}{(1+\lambda x)}$

$$\therefore \frac{d(v^2)}{dx} + \frac{2\lambda}{1+\lambda x}(v^2) = 2g$$

c Let $v^2 = Y$

$$\frac{dY}{dx} + \frac{2\lambda Y}{1+\lambda x} = 2g$$

Use the substitution if you need to. You can solve the equation keeping the v^2 if you wish.

$$\text{Integrating factor} = e^{\int \frac{2\lambda}{1+\lambda x} dx}$$

$$= e^{2\ln(1+\lambda x)}$$

$$= e^{\ln(1+\lambda x)^2} = (1+\lambda x)^2$$

$$\therefore (1+\lambda x)^2 \frac{dY}{dx} + (1+\lambda x)2\lambda Y = 2(1+\lambda x)^2 g$$

$$\frac{d}{dx}\left[(1+\lambda x)^2 Y\right] = 2(1+\lambda x)^2 g$$

$$(1+\lambda x)^2 Y = \frac{2}{3}(1+\lambda x)^3 \times \frac{g}{\lambda} + c$$

$$v^2 = \frac{2g}{3\lambda}(1+\lambda x) + \frac{c}{(1+\lambda x)^2}$$

$Y = v^2$

$$x = 0, v = 0 \Rightarrow 0 = \frac{2g}{3\lambda} + c$$

$$\therefore v^2 = \frac{2g}{3\lambda}(1+\lambda x) - \frac{2g}{3\lambda(1+\lambda x)^2}$$

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Review Exercise 1
Exercise A, Question 40

Question:

A rocket-driven car propels itself forwards in a straight line on a horizontal track by ejecting burnt fuel backwards at a constant rate λ kg s^{-1} and at a constant speed U m s^{-1} relative to the car. At time t seconds, the speed of the car is v m s^{-1} and the total resistance to the motion of the car has magnitude kv N, where k is a positive constant. When $t = 0$ the total mass of the car, including fuel, is M kg. Assuming that at time t seconds some fuel remains in the car,

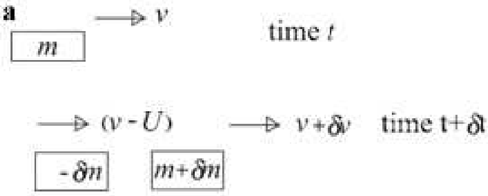
a show that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda U - kv}{M - \lambda t},$$

b find the speed of the car at time t seconds, given that it starts from rest when $t = 0$ and that $\lambda = k = 10$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-U)-mv = -kv\delta t$$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m+U\delta m-mv = -kv\delta t$$

Change of momentum = impulse from resistance

Let $\delta t \to 0$

$\dfrac{\delta m}{\delta t}\delta v \to 0$

$$m\frac{dv}{dt}+U\frac{dm}{dt} = -kv$$

$$t=0, m=M \therefore m = M-\lambda t$$

$$\frac{dm}{dt} = -\lambda$$

The fuel is ejected at a constant rate λ kg s^{-1}.

$$\therefore (M-\lambda t)\frac{dv}{dt}-\lambda U = -kv$$

$$(M-\lambda t)\frac{dv}{dt} = \lambda U - kv$$

$$\frac{dv}{dt} = \frac{\lambda U - kv}{(M-\lambda t)}$$

b $\dfrac{dv}{dt} = \dfrac{10(U-v)}{M-10t}$

$\lambda = k = 10$ in **b**.

$$\int \frac{dv}{U-v} = 10\int \frac{dt}{M-10t}$$

$$-\ln(U-v) = -\ln(M-10t)+c$$

$$t=0,\ v=0\quad c=\ln M - \ln U$$

$$\therefore \ln(U-v) = \ln(M-10t)-\ln M+\ln U$$

$$\ln(U-v) = \ln\left[\frac{U(M-10t)}{M}\right]$$

$$\therefore U-v = \frac{U(M-10t)}{M}$$

$$UM-Mv = UM-10tU$$

$$v = \frac{10Ut}{M}$$

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Review Exercise 1
Exercise A, Question 41

Question:

A rocket-driven car moves along a straight horizontal road. The car has total initial mass M. It propels itself forwards by ejecting mass backwards at a constant rate λ per unit time at a constant speed U relative to the car. The car starts from rest at time $t=0$. At time t the speed of the car is v. The total resistance to motion is modelled as having magnitude kv, where k is a constant.

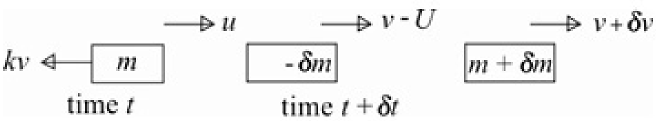
Given that $t < \dfrac{M}{\lambda}$, show that

a $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\lambda U - kv}{M - \lambda t}$,

b $v = \dfrac{\lambda U}{k}\left\{1 - \left(1 - \dfrac{\lambda t}{M}\right)^{\frac{k}{\lambda}}\right\}$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-U)-mv = -kv\delta t$$ ← Change of momentum = impulse

$$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + U\delta m - mv = -kv\delta t$$

Let $\delta t \to 0$ ← $\dfrac{\delta m}{\delta t}\delta v \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + U\frac{\mathrm{d}m}{\mathrm{d}t} = -kv$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\lambda \Rightarrow m = -\lambda t + c$$ ← Mass is ejected at a constant rate λ per unit time.

$$t=0,\ m=M \therefore\ c = M$$

$$\therefore\ m = M - \lambda t$$

$$\therefore (M-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t} - \lambda U = -kv$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda U - kv}{M - \lambda t}$$

b $\int \frac{dv}{\lambda U - kv} = \int \frac{dt}{M - \lambda t}$

Separate the variables, and integrate.

$$-\frac{1}{k}\ln(\lambda U - kv) = -\frac{1}{\lambda}\ln(M - \lambda t) + \ln A$$

$$t = 0, v = 0 \Rightarrow -\frac{1}{k}\ln \lambda U = -\frac{1}{\lambda}\ln M + \ln A$$

$$\ln A = \ln \frac{M^{\frac{1}{\lambda}} - \ln(\lambda U)^{\frac{1}{k}}}{\frac{1}{\lambda}}$$

$$A = \frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}}$$

$$\therefore \ln\left(\frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}}\right) = \ln\left[\frac{(M - \lambda t)^{\frac{1}{\lambda}}}{(\lambda U - kv)^{\frac{1}{k}}}\right]$$

$$\frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}} = \frac{(M - \lambda t)^{\frac{1}{\lambda}}}{(\lambda U - kv)^{\frac{1}{k}}}$$

$$\frac{M^{\frac{k}{\lambda}}}{\lambda U} = \frac{(M - \lambda t)^{\frac{k}{\lambda}}}{\lambda U - kv}$$

Raise each side to the power k, so the power is the same as in the required result.

$$\lambda U - kv = \lambda U\left[\frac{M - \lambda t}{M}\right]^{\frac{k}{\lambda}}$$

$$kv = \lambda U\left[1 - \left(\frac{M - \lambda t}{M}\right)^{\frac{k}{\lambda}}\right]$$

$$v = \frac{\lambda U}{k}\left[1 - \left(1 - \frac{\lambda t}{M}\right)^{\frac{k}{\lambda}}\right]$$

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Review Exercise 2
Exercise A, Question 1

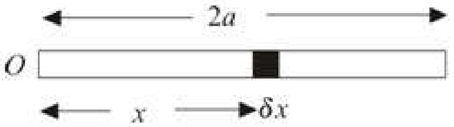
Question:

a Prove, using integration, that the moment of inertia of a uniform rod, of mass m and length $2a$, about an axis perpendicular to the rod through one end is $\frac{4}{3}ma^2$.

b Hence, or otherwise, find the moment of inertia of a uniform square lamina, of mass M and side $2a$, about an axis through one corner and perpendicular to the plane of the lamina. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{2a}$

Consider an element of length δx at a distance x from one end of the rod O.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{2a}\delta x\right)x^2 = \frac{mx^2}{2a}\delta x$$

The mass of the element is its length (δx) multiplied by the mass per unit length $\left(\frac{m}{2a}\right)$.

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{2a}\delta x$$

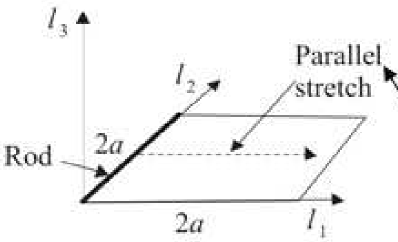
As $\delta x \to 0$

$$I = \int_0^{2a} \frac{mx^2}{2a} \mathrm{d}x = \frac{m}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{m}{2a}\left[\frac{8a^3}{3} - 0\right] = \frac{4}{3}ma^2, \text{ as required}$$

As the element ranges from O to the other end of the rod, x ranges from 0 to $2a$. So 0 and $2a$ are the limits of integration.

b



Let l_1 and l_2 be axes along two of the sides of the square and l_3 be the axis through the corner perpendicular to l_1 and l_2, as shown in this sketch. The question asks you to find the moment of inertia about l_3.

By the stretching rule, the moment of inertia of the lamina about l_1 and l_2 is given by

$$I_{l_1} = I_{l_2} = \frac{4}{3}ma^2$$

As the lamina can be formed by taking a rod and stretching it parallel to the axis l_1, without altering the distribution of the mass relative to l_1, then the moment of inertia of the lamina about l_1 is the same as the rod, $\frac{4}{3}ma^2$.

By the perpendicular axes theorem

$$I_{l_3} = I_{l_1} + I_{l_2} = \frac{4}{3}ma^2 + \frac{4}{3}ma^2 = \frac{8}{3}ma^2$$

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Review Exercise 2
Exercise A, Question 2

Question:

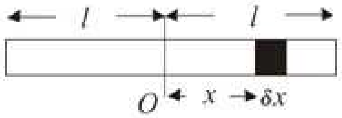
a Show, using integration, that the moment of inertia of a uniform rod, of length $2l$ and mass m, about an axis through its centre and perpendicular to the rod is $\frac{1}{3}ml^2$.

A uniform square plate, of mass M, has edges of length $2a$.

b Find the moment of inertia of the plate about an axis through its centre perpendicular to the plane of the plate. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{2l}$

When proving results you usually need to know the 'density' of the object, here the mass per unit length.

Consider an element of length δx at a distance x from the middle of the rod O.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{2l}\delta x\right)x^2 = \frac{mx^2}{2l}\delta x$$

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{2l}\delta x$$

The whole rod is the sum of the small pieces.

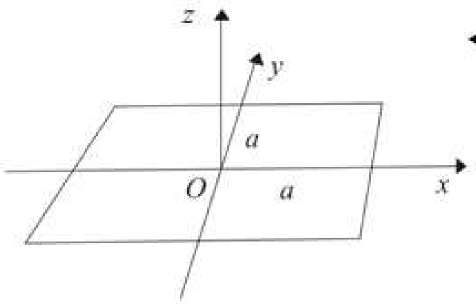
As $\delta x \to 0$

$$I = \int_{-l}^{l} \frac{mx^2}{2l}\,\mathrm{d}x = \frac{m}{2l}\left[\frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{m}{2l}\left[\frac{l^3}{3} - \left(-\frac{l^3}{3}\right)\right] = \frac{1}{3}ml^2, \text{ as required}$$

As the small pieces range from one end of the rod to the other, x ranges from $-l$ at one end to l at the other. So $-l$ and l are the limits of the definite integral.

b



Let O be the centre of the plate, Ox and Oy be axes parallel to the sides of the square and Oz be the axis through O perpendicular to the plate, as shown in this sketch. The question asks you to find the moment of inertia about Oz.

By the stretching rule, the moment of inertia of the lamina about Ox and Oy is given by

$$I_{Ox} = I_{Oy} = \frac{1}{3}ma^2$$

By symmetry, the moment of inertia about Ox and Oy is the same.

By the perpendicular axes theorem

$$I_{Oz} = I_{Ox} + I_{Oy} = \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

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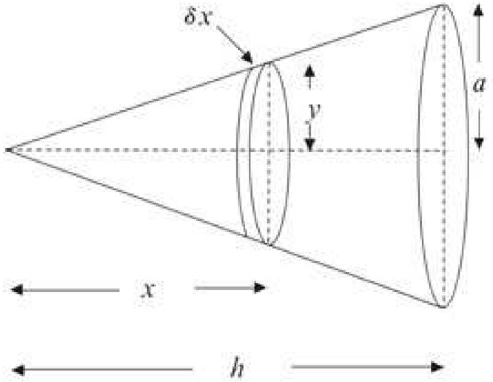
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 3

Question:

Given that the moment of inertia of a uniform disc, of mass m and radius r, about an axis through the centre perpendicular to the disc is $\frac{1}{2}mr^2$, show by integration that the moment of inertia of a uniform solid circular cone, of base radius a, height h and mass M, about its axis of symmetry is $\frac{3}{10}Ma^2$. ***E***

Solution:



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{M}{\frac{1}{3}\pi a^2 h} = \frac{3M}{\pi a^2 h}$$

You are expected to know that the volume of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.

The moment of inertia of an elementary disc about the axis of symmetry is given by

$$\delta I = \frac{1}{2}(\delta m)y^2$$

This is the point at which you use the given result, that the moment of inertia of a disc is $\frac{1}{2}mr^2$. The mass of the disc is δm and the radius y.

By similar triangles

$$\frac{y}{x} = \frac{a}{h} \Rightarrow y = \frac{ax}{h}$$

Hence

$$\delta I = \frac{1}{2}(\pi y^2 \delta x)\left(\frac{3M}{\pi a^2 h}\right)y^2 = \frac{3M}{2a^2 h} y^4 \delta x$$

$$= \frac{3M}{2a^2 h}\left(\frac{ax}{h}\right)^4 \delta x = \frac{3Ma^2}{2h^5} x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit length, $\frac{3M}{\pi a^2 h}$.

For the complete cone

$$I = \sum \delta I = \sum \frac{3Ma^2}{2h^5} x^4 \delta x$$

As $\delta x \to 0$

$$I = \int_0^h \frac{3Ma^2}{2h^5} x^4 \, dx = \frac{3Ma^2}{2h^5}\left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{3Ma^2}{2h^5}\left(\frac{h^5}{5} - 0\right) = \frac{3}{10}Ma^2, \text{ as required}$$

As the thin, or elementary, discs range from the vertex to the base, x ranges from 0 to h. So 0 and h are the limits of integration.

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Review Exercise 2
Exercise A, Question 4

Question:

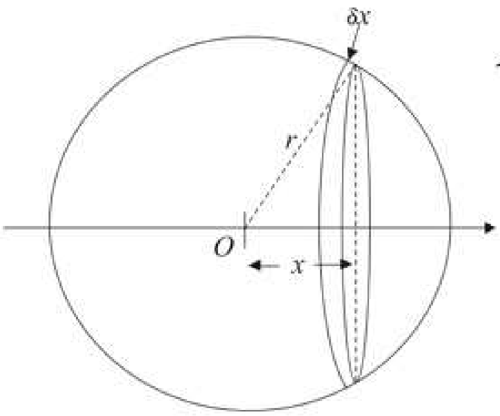
a Prove, using integration, that the moment of inertia of a uniform solid sphere, of mass M and radius r, about a diameter is $\frac{2}{5}Mr^2$.

[You may assume that the moment of inertia of a uniform disc, of mass m and radius a, about an axis through the centre perpendicular to the disc is $\frac{1}{2}ma^2$.]

b Hence obtain the moment of inertia of a solid hemisphere, of mass m and radius r, about a diameter of its plane face. ***E***

Solution:

a



You consider the cone to be made up of thin discs, each of thickness δx, with the centre of the disc at a distance x from the centre O of the sphere. If the radius of the disc is y, then, using the formula $V = \pi r^2 h$ for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the sphere is

$$\frac{M}{\frac{4}{3}\pi r^3} = \frac{3M}{4\pi r^3}$$

You are expected to know that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

The moment of inertia of an elementary disc is given by

$$\delta I = \frac{1}{2}(\delta m) y^2$$

$$y^2 = r^2 - x^2$$

You know, from module C2, that the equation of the circle is $x^2 + y^2 = r^2$.

Hence

$$\delta I = \frac{1}{2}\left(\pi y^2 \delta x\right)\left(\frac{3M}{4\pi r^3}\right) y^2 = \frac{3M}{8r^3} y^4 \delta x$$

$$= \frac{3M}{8r^3}\left(r^2 - x^2\right)^2 \delta x = \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right)\delta x$$

For the complete sphere

$$I = \sum \delta I = \sum \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right)\delta x$$

As $\delta x \to 0$

$$I = \int_{-r}^{r} \frac{3M}{8r^3}\left(r^4 - 2r^2x^2 + x^4\right)\mathrm{d}x$$

A common error is to integrate r^4 as $\frac{r^5}{5}$. With respect to x, r is a constant so $\int r^4 \,\mathrm{d}x = r^4 x$.

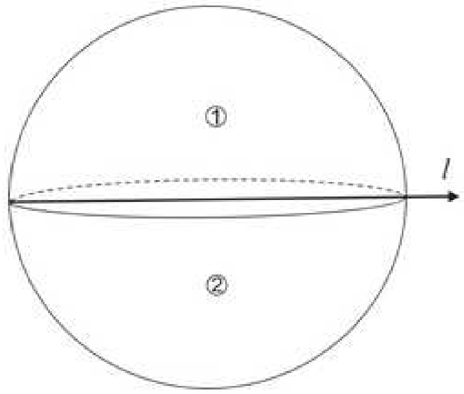
$$= \frac{3M}{8r^3}\left[r^4 x - \frac{2r^2x^3}{3} + \frac{x^5}{5}\right]_{-r}^{r}$$

$$= \frac{3M}{8r^3}\left[\left(r^5 - \frac{2r^5}{3} + \frac{r^5}{5}\right) - \left(-r^5 + \frac{2r^5}{3} - \frac{r^5}{5}\right)\right]$$

$$= \frac{3M}{8r^3} \times 2r^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{3Mr^2}{4} \times \frac{8}{15}$$

$$= \frac{2}{5}Mr^2, \text{ as required}$$

b



If you consider a complete sphere as being made up to two hemispheres, then, by the addition rule, the sum of the moments of inertia of the two hemispheres about the axis *l*, in this diagram, must equal the moment of inertia of the whole sphere about *l*. So a hemisphere has half the moment of inertia of the whole sphere. However, the mass m is not now the mass of the whole sphere and you must be careful to avoid the incorrect answer $\frac{1}{5}mr^2$.

If the mass of the whole sphere is $2m$ and the radius of the sphere is r, then using the result of part **a**, the moment of inertia of the whole sphere is

$$\frac{2}{5}(2m)r^2 = \frac{4}{5}mr^2$$

By symmetry, the moment of inertia of the hemisphere, labelled ① in the diagram about *l*, must equal the moment of inertia of the hemisphere, labelled ② in the diagram, about the same axis.

Hence, the moment of inertia of one hemisphere is

$$\frac{1}{2}\times\frac{4}{5}mr^2 = \frac{2}{5}mr^2.$$

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Edexcel AS and A Level Modular Mathematics

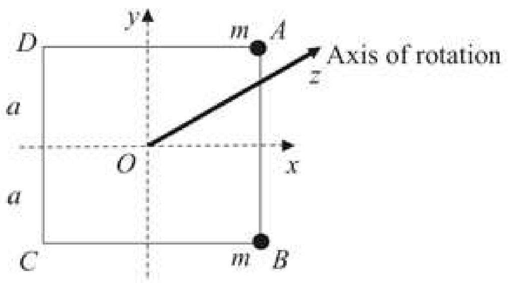
Review Exercise 2
Exercise A, Question 5

Question:

A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about an axis through its centre O. Particles, each of mass m, are attached at the points A and B. The system is released from rest with AB vertical.

Show that the angular speed of the square when AB is horizontal is $\sqrt{\left(\frac{6g}{7a}\right)}$. ***E***

Solution:



The moment of inertia of the lamina alone about the axis of rotation is given, using the perpendicular axes theorem, by

$$I_{Oz} = I_{Ox} + I_{Oy}$$
$$= \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$
$$OA^2 = OB^2 = a^2 + a^2 = 2a^2$$

> The moment of inertia of the lamina about Ox in the diagram is, by the stretching rule, the same as the moment of inertia of a uniform rod about its centre.

The moment of inertia of the lamina together with the particles about the axis of rotation is given by

$$I = I_{Oz} + m\left(OA^2\right) + m\left(OB^2\right)$$
$$= \frac{2}{3}ma^2 + m\left(2a^2\right) + m\left(2a^2\right) = \frac{14}{3}ma^2$$

> In many questions, involving moments of inertia, you need to begin by finding the moment of inertia of the whole system, in this case the lamina with both particles, about the axis of rotation.

As the loaded plate rotates from the position with AB vertical to the position with AB horizontal
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mg \times 2a$$
$$\dot{\theta}^2 = \frac{4mga}{I} = \frac{4mga}{\frac{14}{3}ma^2} = \frac{6g}{7a}$$
$$\dot{\theta} = \sqrt{\left(\frac{6g}{7a}\right)}, \text{ as required}$$

> As AB moves from the vertical to the horizontal, the position of the centre of the lamina is unchanged and the level of the particle at B is the same in the vertical and horizontal positions. So the only potential energy lost is by the particle at A falling a distance $2a$.

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Review Exercise 2
Exercise A, Question 6

Question:

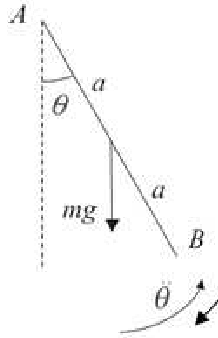
A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a smooth horizontal axis through A and perpendicular to the plane. The rod hangs in equilibrium with B below A. The rod is rotated through a small angle and released from rest at time $t = 0$.

a Show that the motion is approximately simple harmonic.

b Using this approximation, find the time t when the rod is first vertical after being released. ***E***

Solution:

a



Rates of change of the angle are always measured in the direction of θ increasing. So here the moment of the mass of the rod about A will be negative.

The equation of rotational motion about A is

$$L = I\ddot{\theta}$$

$$-mga\sin\theta = \frac{4}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3g}{4a}\sin\theta$$

Standard results for moments of inertia are given in the Formulae Booklet.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{3g}{4a}\theta$$

Comparing this equation with $\ddot{\theta} = -\omega^2\theta$, $\omega^2 = \frac{3g}{4a}$ and you will need this to answer part **b**.

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic.

b

$$t = \frac{1}{4}T = \frac{1}{4}\times\frac{2\pi}{\omega} = \frac{\pi}{2\omega}$$

$$= \frac{\pi}{2\sqrt{\left(\frac{3g}{4a}\right)}} = \pi\sqrt{\left(\frac{a}{3g}\right)}$$

The period of the motion is the time it takes for the rod, after it is released, to return to its starting position for the first time. The time from the first release to when the rod first reaches the vertical is one quarter of this period.

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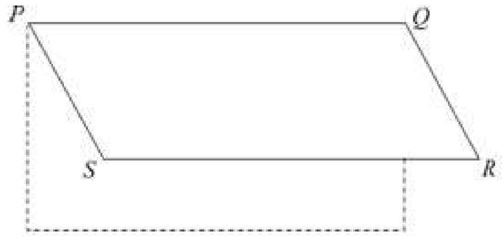
Review Exercise 2
Exercise A, Question 7

Question:

A uniform lamina of mass m is in the shape of a rectangle $PQRS$, where $PQ = 8a$ and $QR = 6a$.

a Find the moment of inertia of the lamina about the edge PQ.



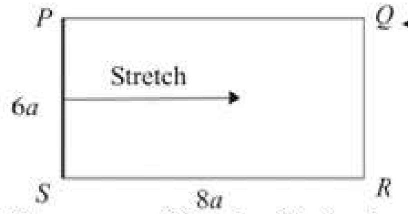
The flap on a letterbox is modelled as such a lamina. The flap is free to rotate about an axis along its horizontal edge PQ, as shown in the figure. The flap is released from rest in a horizontal position. It then swings down into a vertical position.

b Show that the angular speed of the flap as it reaches the vertical position is $\sqrt{\left(\frac{g}{2a}\right)}$.

c Find the magnitude of the vertical component of the resultant force of the axis PQ on the flap, as it reaches the vertical position. ***E***

Solution:

a

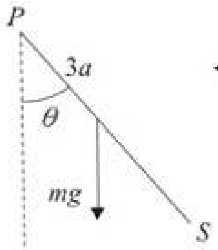


The moment of inertia of the lamina about PQ is given by

$$I = \frac{4}{3}ml^2 = \frac{4}{3}m(3a)^2 = 12ma^2$$

As the lamina $PQRS$ can be formed by stretching a rod PS without altering the distribution of the mass relative to PQ, then, by the stretching rule, the moment of inertia of the lamina about PQ is the same as the rod, $\frac{4}{3}ml^2$, where $2l$ is the length of PS. Here $PS = 2l = 6a$.

b



This diagram is drawn viewing the flap from the side. The weight acts at the centre of mass of the lamina.

Let PS make an angle θ with the downward vertical at time t.
When PS reaches the vertical
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mg \times 3a$$

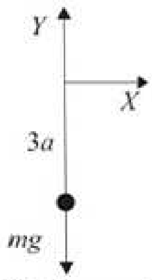
$$6ma^2\dot{\theta}^2 = 3mga$$

$$\dot{\theta}^2 = \frac{3mga}{6ma^2} = \frac{g}{2a}$$

In falling from the horizontal to the vertical position, the centre of mass of the lamina falls a distance $3a$.

The angular speed of the flap as it reaches the vertical position is , $\sqrt{\left(\frac{g}{2a}\right)}$ as required.

c



When writing down an equation of motion, you consider the whole mass of the lamina to be at the centre of mass of the lamina, which is $3a$ from the axis of rotation.

Let the magnitude of the vertical component of the resultant force of the axis PQ on the flap, as it reaches the vertical position be Y.

$$\text{R}(\uparrow) \quad \mathbf{F} = m\mathbf{a}$$

$$Y - mg = mr\dot{\theta}^2$$

$$= m(3a)\frac{g}{2a}$$

$$Y = mg + \frac{3}{2}mg = \frac{5}{2}mg$$

From part **b**, we know that $\dot{\theta}^2 = \frac{g}{2a}$.

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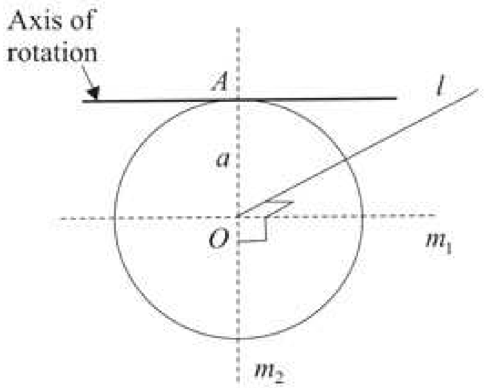
Review Exercise 2
Exercise A, Question 8

Question:

A uniform circular disc has mass m and radius a. The disc can rotate freely about an axis that is in the same plane as the disc and tangential to the disc at a point A on its circumference. The disc hangs at rest in equilibrium with its centre O vertically below A.
A particle P of mass m is moving horizontally and perpendicular to the disc with speed $\sqrt{(kga)}$, where k is a constant. The particle then strikes the disc at O and adheres to it at O.
Given that the disc rotates through an angle of 90° before first coming to instantaneous rest, find the value of k. ***E***

Solution:



By symmetry, the moments of inertia about the perpendicular axes m_1 and m_2, shown in the diagram, are equal.

By the perpendicular axes theorem,

$$I_l = I_{m_1} + I_{m_2}$$

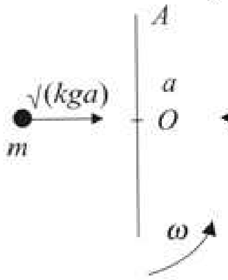
$$\frac{1}{2}ma^2 = 2I_{m_1}$$

$$I_{m_1} = \frac{1}{4}ma^2$$

The standard result for a disc, of mass m and radius a, is that the moment of inertia of the disc about an axis, shown in this diagram as l, is $\frac{1}{2}ma^2$. From this, you find the moment of inertia about the axis of rotation using both the perpendicular and parallel axes theorems.

Using the parallel axes theorem, the moment of inertia of the disc, I_d, about the axis of rotation is given by

$$I_d = I_{m_1} + ma^2 = \frac{1}{4}ma^2 + ma^2 = \frac{5}{4}ma^2$$



This sketch is drawn looking at the disc edge on.

Let the angular velocity of the disc and P immediately after impact be ω.
Conservation of angular momentum about A

$$mva = I\dot{\theta}$$

$$m\sqrt{(kga)}a = \left(\frac{5}{4}ma^2 + ma^2\right)\omega$$

$$m\sqrt{(kga)}a = \frac{9}{4}ma^2\omega$$

$$\omega = \frac{4}{9}\sqrt{\left(\frac{kg}{a}\right)}$$

By the addition rule, the moment of inertia of the disc and P about the axis of rotation is the moment of inertia of the disc $\left(\frac{5}{4}ma^2\right)$ added to the moment of inertia of P about the axis of rotation (ma^2).

Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg \times a$$

$$\frac{1}{2} \times \frac{9}{4}ma^2 \times \frac{16kg}{81a} = 2mga$$

$$\frac{2}{9}kmga = 2mga$$

$$k = \frac{18mga}{2mga} = 9$$

In moving from the vertical through $90°$ both the centre of mass of the disc and P, that is a total mass of $2m$, rise the distance a.

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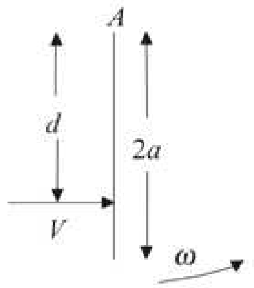
Review Exercise 2
Exercise A, Question 9

Question:

A rod AB, of length $2a$ and mass $2m$, lies at rest on a smooth horizontal table and is pivoted about a smooth vertical axis through A. A small body of mass m, moving on the table with speed V at right angles to the rod, strikes the rod at a distance d from A. Given that the body sticks to the rod after impact, find the angular speed with which the rod starts to move. ***E***

Solution:



Let the angular speed of the rod and body about A immediately after the impact be ω.
Conservation of angular momentum about A

$mVd = I\omega$

Before the impact, the angular momentum of the body of mass m is the linear momentum of the body (mV) multiplied by the distance d.

$$mVd = \left(\frac{4}{3}(2m)a^2 + md^2\right)\omega$$

As the mass of the particle is $2m$, the moment of inertia of the rod about its end is $\frac{4}{3}(2m)a^2$. To this you must add the moment of inertia of the body of mass m about A.

$$\omega = \frac{mVd}{\frac{8}{3}ma^2 + md^2}$$

$$= \frac{3Vd}{8a^2 + 3d^2}$$

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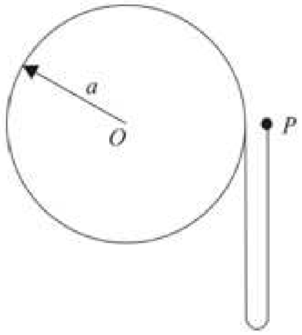
Review Exercise 2
Exercise A, Question 10

Question:



The figure shows a pulley in the form of a uniform disc of mass $2m$, centre O, and radius a. The pulley is free to rotate in a vertical plane about a fixed smooth horizontal axis through O. A light inextensible string has one end attached to a point on the rim of the pulley and is wrapped several times round the rim. The portion of the string which is not wrapped round the pulley is of length $4a$ and has a particle P of mass m attached to its free end. P is held close to the rim of the disc and level with O, with the disc at rest. The particle P is released from rest in this position.
Determine the angular speed of the disc immediately after the string becomes taut. ***E***

Solution:

Let v be the speed P immediately before the string becomes taut.

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times g \times 4a$

$v = \sqrt{(8ga)}$

(P falls a distance $4a$ freely under gravity before the string becomes taut.)

The combined moment of inertia of the pulley and P about O is given by

$$I = \frac{1}{2}(2m)a^2 + ma^2 = 2ma^2$$

Let the angular speed of the pulley about O immediately after the impact be ω.

(ω will also be the angular speed of P about O.)

Conservation of angular momentum about O

$$mva = I\dot{\theta}$$

$$m \times \sqrt{(8ag)} \times a = 2ma^2\omega$$

(Before the impulse, use the moment of the linear momentum of P about O. After the impulse, use $I\omega$ for the disc and P.)

$$\omega = \frac{\sqrt{(8ag)}}{2a} = \sqrt{\left(\frac{2g}{a}\right)}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 11

Question:

A uniform circular disc, of mass m and radius r, has a diameter AB. The point C on AB is such that $AC = \frac{1}{2}r$. The disc can rotate freely in a vertical plane about a horizontal axis through C, perpendicular to the plane of the disc. The disc makes small oscillations in a vertical plane about the position of equilibrium in which B is below A.

a Show that the motion is approximately simple harmonic.

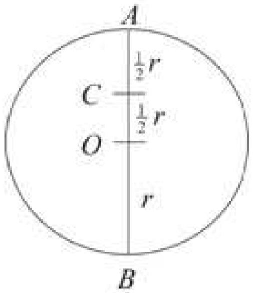
b Show that the period of this approximate simple harmonic motion is $\pi\sqrt{\frac{6r}{g}}$.

The speed of B when it is vertically below A is $\sqrt{\frac{gr}{54}}$. The disc comes to rest when CB makes an angle α with the downward vertical.

c Find an approximate value of α. ***E***

Solution:

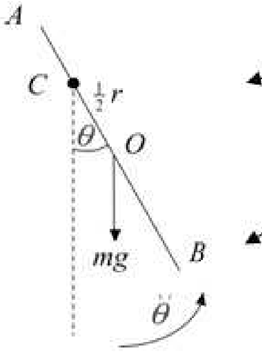
a



Let the centre of the disc be O.
By the parallel axes theorem, the moment of inertia, I_C, of the disc about C is given by

$$I_C = I_O + m\left(\frac{1}{2}r\right)^2$$

$$= \frac{1}{2}mr^2 + \frac{1}{4}mr^2 = \frac{3}{4}mr^2$$



This diagram is drawn looking at the disc edge on.

Rates of change of the angle are always measured in the direction of θ increasing. So here the moment of the mass of the rod about A will be negative.

The equation of rotational motion about C is

$$L = I\ddot{\theta}$$

$$-mg\left(\frac{1}{2}r\sin\theta\right) = \frac{3}{4}mr^2\ddot{\theta}$$

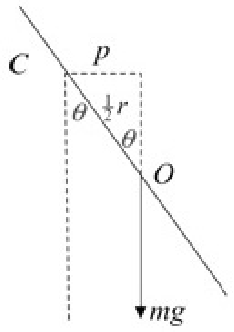
$$\ddot{\theta} = -\frac{2g}{3r}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{2g}{3r}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{2g}{3r}$.



The perpendicular distance from C to the line of action of the weight is given by

$\frac{p}{\frac{1}{2}r} = \sin\theta \Rightarrow p = \frac{1}{2}r\sin\theta$, so the magnitude of moment of the weight about C is $mg \times \left(\frac{1}{2}r\sin\theta\right)$. This moment is tending to make θ decrease, so it has a negative sign in this equation.

b The period of approximate simple harmonic motion is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{2g}{3r}\right)}} = 2\pi\sqrt{\left(\frac{3r}{2g}\right)}$$

$$= \pi\sqrt{\left(\frac{6r}{g}\right)}, \text{ as required}$$

c The speed of P at B is the maximum speed during the simple harmonic motion.

Using $v = r\dot{\theta}$ with the speed of B

$$\sqrt{\left(\frac{gr}{54}\right)} = \left(\frac{3}{2}r\right)\dot{\theta}$$

$$\dot{\theta} = \sqrt{\left(\frac{gr}{54}\right)} \times \frac{2}{3r} = \frac{2}{3}\sqrt{\left(\frac{g}{54r}\right)}$$

At the maximum angular speed

$$\dot{\theta} = \omega\alpha$$

In module M3, you learnt that the maximum speed during simple harmonic motion is at the centre of the motion and is given by $v = \omega a$, where a is the amplitude. $\dot{\theta} = \omega\alpha$ is the corresponding formula for angular motion. α is the greatest angle during the motion, which is where the body is instantaneously at rest.

$$\frac{2}{3}\sqrt{\left(\frac{g}{54r}\right)} = \sqrt{\left(\frac{2g}{3r}\right)}\alpha$$

$$\alpha = \frac{2}{3}\sqrt{\left(\frac{1}{54}\right)} \times \sqrt{\left(\frac{3}{2}\right)} = \frac{2}{3}\sqrt{\left(\frac{3}{108}\right)}$$

$$= \frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$$

This is an approximate answer. There are other methods of solving this question, for example using energy, which would give slightly different answers, but answers should all approximate to 0.11 radians.

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Review Exercise 2
Exercise A, Question 12

Question:

A uniform circular disc, of mass m, radius a and centre O, is free to rotate in a vertical plane about a fixed smooth horizontal axis. The axis passes through the mid-point A of a radius of the disc.

a Find an equation of motion for the disc when the line AO makes an angle θ with the downward vertical through A.

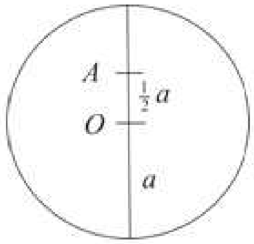
b Hence find the period of small oscillations of the disc about its position of stable equilibrium.

When the line AO makes an angle θ with the downward vertical through A, the force acting on the disc at A is $\mathbf{F}$.

c Find the magnitude of the component of $\mathbf{F}$ perpendicular to AO. ***E***

Solution:

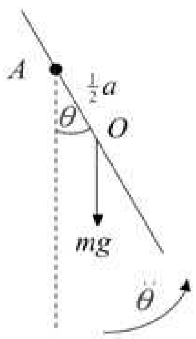
a



By the parallel axes theorem, the moment of inertia, I_A, of the disc about A is given by

$$I_A = I_O + m\left(\frac{1}{2}a\right)^2$$

$$= \frac{1}{2}ma^2 + \frac{1}{4}ma^2 = \frac{3}{4}ma^2$$



This diagram is drawn looking at the disc edge on.

The equation of motion about A is

$$L = I\ddot{\theta}$$

$$-mg\left(\frac{1}{2}a\sin\theta\right) = \frac{3}{4}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{2g}{3a}\sin\theta$$

The moment of the weight about A is tending to make θ decrease so this term has a negative sign.

b For small θ, $\sin\theta \approx \theta$

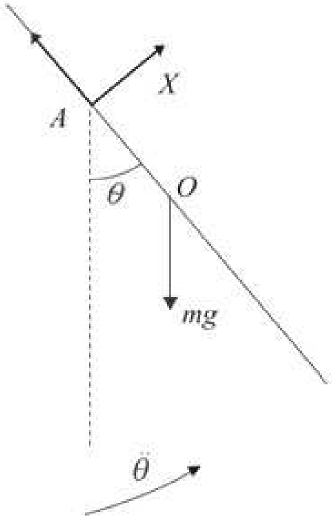
Hence

$$\ddot{\theta} = -\frac{2g}{3a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{2g}{3a}$. The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\left(\frac{2g}{3a}\right)}} = 2\pi\sqrt{\left(\frac{3a}{2g}\right)}$$

c



Let the component of **F** perpendicular to AO be X.

R($\perp AO$)

$$X - mg\sin\theta = mr\ddot{\theta}$$

$$= m\left(\frac{a}{2}\right)\left(-\frac{2g}{3a}\sin\theta\right) = -\frac{1}{3}mg\sin\theta$$

$$X = \frac{2}{3}mg\sin\theta$$

In part **c**, unlike part **b**, you are not told that the oscillations are small, so you must use the result of part **a** to substitute for $\ddot{\theta}$.

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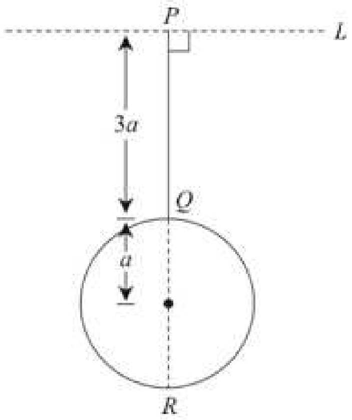
Review Exercise 2
Exercise A, Question 13

Question:



A thin uniform rod PQ has mass m and length $3a$. A thin uniform circular disc, of mass m and radius a, is attached to the rod at Q in such a way that the rod and the diameter QR of the disc are in a straight line with $PR = 5a$. The rod together with the disc form a composite body, as shown in the figure. The body is free to rotate about a fixed smooth horizontal axis L through P, perpendicular to PQ and in the plane of the disc.

a Show that the moment of inertia of the body about L is $\frac{77ma^2}{4}$.

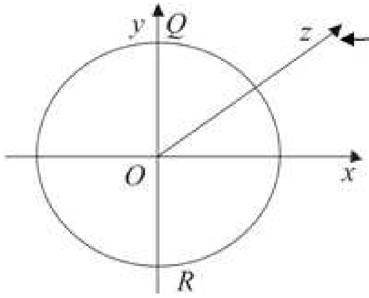
When PR is vertical, the body has angular speed ω and the centre of the disc strikes a stationary particle of mass $\frac{1}{2}m$. Given that the particle adheres to the centre of the disc,

b find, in terms of ω, the angular speed of the body immediately after the impact.

E

Solution:

a



The standard result for the moment of inertia of a disc $\left(I = \frac{1}{2}ma^2\right)$ is for an axis through its centre perpendicular to the plane of the disc. From this you need to find the moment of inertia about a diameter. You then use the moment of inertia about a diameter to find the moment of inertia of the disc about L.

Let O be the centre of the disc.
By the perpendicular axes theorem,
the moment of inertia, I_{Ox}, about the diameter through O perpendicular to PR is given by

$$I_{Ox} + I_{Oy} = I_{Oz}$$
$$2I_{Ox} = \frac{1}{2}ma^2$$
$$I_{Ox} = \frac{1}{4}ma^2$$

By the parallel axes theorem the moment of inertia of the disc, I_d, about L is given by

$$I_d = I_{Ox} + m(4a)^2$$
$$= \frac{1}{4}ma^2 + 16ma^2 = \frac{65ma^2}{4}$$

The parallel axes are Ox and L. The distance between these axes is $OP = 4a$.

The moment of inertia of the body about L is given by

$$I = I_d + \frac{4}{3}m\left(\frac{3a}{2}\right)^2$$
$$= \frac{65ma^2}{4} + 3ma^2 = \frac{77ma^2}{4}\text{, as required}$$

Using the standard result for the moment of inertia of a rod about its end $\left(I = \frac{4}{3}ml^2\right)$. The length of the rod PQ is $3a$ giving $2l = 3a \Rightarrow l = \frac{3}{2}a$.

b The moment of inertia, I', of the body and the particle combined is given by

$$I' = I + \frac{1}{2}m(4a)^2$$
$$= \frac{77ma^2}{4} + 8ma^2 = \frac{109ma^2}{4}$$

The particle of mass $\frac{1}{2}m$ is at O, that is $4a$ from L.

Let the angular speed of the body immediately after the impact be ω'.
Conservation of angular momentum about L

$$I'\omega' = I\omega$$
$$\frac{109ma^2}{4}\omega' = \frac{77ma^2}{4}\omega$$
$$\omega' = \frac{77}{109}\omega$$

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Review Exercise 2
Exercise A, Question 14

Question:

A thin uniform rod AB, of mass M and length $2L$, is freely pivoted at A. The rod hangs vertically with B below A. A particle of mass $\frac{1}{2}M$, travelling horizontally with speed u, strikes the rod at B. After this impact the particle is at rest and the rod starts to move with angular speed ω.

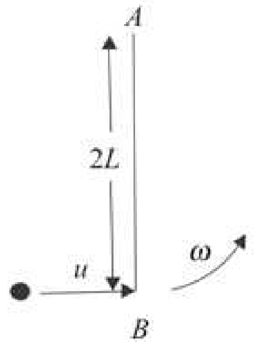
a Show that $\omega = \frac{3u}{4L}$.

The rod comes to instantaneous rest when AB is inclined at an angle $\arccos\left(\frac{1}{3}\right)$ to the downward vertical.

b Find u in terms of L and g. ***E***

Solution:

a



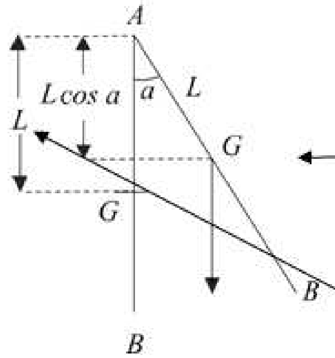
Conservation of angular momentum about A.

$$\frac{1}{2}Mu \times 2L = \frac{4}{3}ML^2\omega$$

$$\omega = \frac{6MLu}{8ML^2} = \frac{3u}{4L}, \text{ as required}$$

After the impact the particle is at rest, so the only angular momentum is that of the rod AB about its end A.

b



In this diagram, G is the centre of mass of the rod and $\alpha = \arccos\left(\frac{1}{3}\right)$.

Conservation of energy

Kinetic energy lost = Potential energy gained

$$\frac{1}{2}I\omega^2 = Mg(L - L\cos\alpha)$$

As the rod swings through an angle α, the centre of mass of the rod G rises a distance $L - L\cos\alpha$.

$$\frac{1}{2} \times \frac{4}{3}ML^2\left(\frac{3u}{4L}\right)^2 = Mg\left(L - \frac{1}{3}L\right)$$

Using the result of part **a**.

$$\frac{3}{8}Mu^2 = \frac{2}{3}MgL$$

$$u^2 = \frac{16}{9}gL$$

$$u = \frac{4}{3}\sqrt{(gL)}$$

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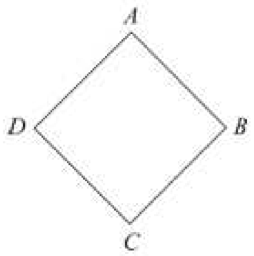
Review Exercise 2
Exercise A, Question 15

Question:



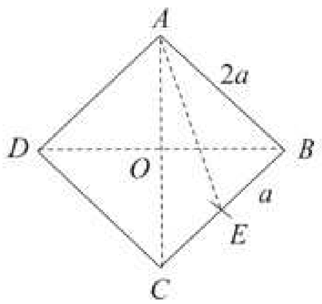
The figure shows four uniform rods, each of mass m and length $2a$, rigidly fixed together to form a square framework $ABCD$. The framework is free to rotate about a fixed smooth horizontal axis which passes through A and is perpendicular to the plane $ABCD$.

a Find the moment of inertia of the framework about this axis.

b Show, that for small oscillations of the framework about its position of equilibrium with C below A, the period of oscillation of the motion is $2\pi\left(\frac{(5\sqrt{2})a}{3g}\right)^{\frac{1}{2}}$.

Solution:

a



Let E be the mid-point of BC.

$AE^2 = a^2 + (2a)^2 = 5a^2$

Using Pythagoras' Theorem.

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{3}ma^2 + mAE^2$$

$$= \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$$

The moment of inertia of the framework about the axis through A is given by

$$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

$$= \frac{4}{3}ma^2 + \frac{16}{3}ma^2 + \frac{16}{3}ma^2 + \frac{4}{3}ma^2$$

$$= \frac{40}{3}ma^2$$

By symmetry, the moment of inertia of the rod CD about the axis through A is the same as the moment of inertia of the rod BC.

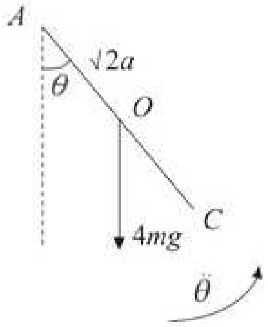
b Let O be the centre of the framework

$AO^2 + BO^2 = AB^2 = 4a^2$

$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2$

$AO = \sqrt{2}a$

The total weight of the framework, $4mg$, acts at the centre of the framework O and, to form the equation of motion, you need to find the distance of the centre of mass from the axis of rotation.



This diagram is drawn looking at the oscillation of the framework from the side.

The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-4mg\left(\sqrt{2}a\right)\sin\theta = \frac{40}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{10a}\sin\theta = -\frac{3g}{5\sqrt{2}a}\sin\theta$$

The moment of the weight about the axis through A is making θ decrease and so has a negative sign in the equation of motion.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{3g}{5\sqrt{2}a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3g}{5\sqrt{2}a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{5\sqrt{2}a}\right)^{\frac{1}{2}}} = 2\pi\left(\frac{5\sqrt{2}a}{3g}\right)^{\frac{1}{2}}$$, as required

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Review Exercise 2
Exercise A, Question 16

Question:

A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through A and is perpendicular to the plane of the lamina. The moment of inertia of the lamina about L is $\frac{8ma^2}{3}$.

Given that the lamina is released from rest when the line AC makes an angle of $\frac{\pi}{3}$ with the downward vertical,

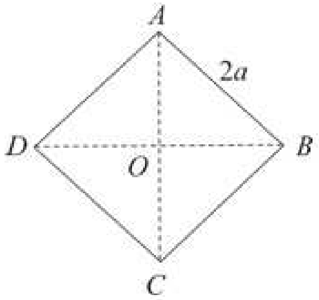
a find the magnitude of the vertical component of the force acting on the lamina at A when the line AC is vertical.

Given instead that the lamina now makes small oscillations about its position of stable equilibrium,

b find the period of these oscillations. ***E***

Solution:

a



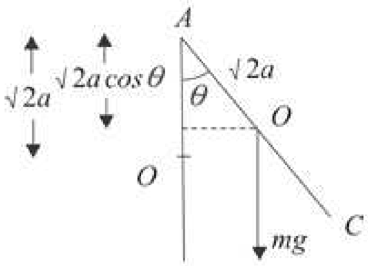
Let O be the centre of mass of the lamina

$$AO^2 + OB^2 = AB^2$$

$$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2$$

$$AO = \sqrt{2}a$$

The weight of the lamina, mg, acts at the centre of the lamina O and you need to find the distance of the centre of mass from the axis of rotation.



As the lamina swings through an angle θ with the vertical, the centre of mass falls a distance of $\sqrt{2}a - \sqrt{2}a\cos\theta$.

Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = mg\left(\sqrt{2}a - \sqrt{2}a\cos\theta\right)$$

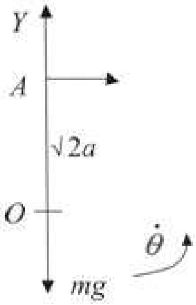
$$\frac{1}{2} \times \frac{8ma^2}{3}\dot{\theta}^2 = mg\sqrt{2}a\left(1 - \cos\frac{\pi}{3}\right) = \frac{mg\sqrt{2}a}{2}$$

$$\dot{\theta}^2 = \frac{3\sqrt{2}g}{8a}$$

The equation of motion needed to find the vertical component will contain a term for the radial acceleration $r\dot{\theta}^2$ and $\dot{\theta}^2$, when AO is vertical, can be found by conservation of energy.

$$1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}$$

When AC is vertical let the vertical component of the force acting on the lamina at A be Y.



R(↑) $\mathbf{F} = m\mathbf{a}$

$$Y - mg = m(\sqrt{2}a)\dot{\theta}^2$$

$$Y = mg + m\sqrt{2}a \times \frac{3\sqrt{2}g}{8a}$$

$$= mg + \frac{3}{4}mg = \frac{7}{4}mg$$

b The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-mg(\sqrt{2}a)\sin\theta = \frac{8}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{8a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{3\sqrt{2}g}{8a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3\sqrt{2}g}{8a}$. The period of small oscillations is given by

> This approximation, in radians, is accurate to within 1% up to 0.24 radians or up to 13.75°. This is accurate enough for many practical purposes.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3\sqrt{2}g}{8a}\right)^{\frac{1}{2}}} = 2\pi\left(\frac{8a}{3\sqrt{2}g}\right)^{\frac{1}{2}} = 2\pi\left(\frac{4\sqrt{2}a}{3g}\right)^{\frac{1}{2}}$$

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Review Exercise 2
Exercise A, Question 17

Question:

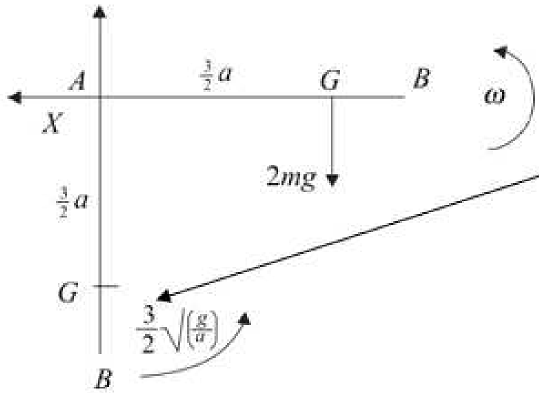
A uniform rod AB, of length $2a$ and mass m, can rotate freely about a fixed horizontal axis through A. A particle of mass m is attached at B. When AB is vertical, with B below A, the system has angular speed $\frac{3}{2}\sqrt{\left(\frac{g}{a}\right)}$.

a Show that, when AB is horizontal, its angular speed is $\frac{3}{4}\sqrt{\left(\frac{2g}{a}\right)}$.

b Find the horizontal component of the force exerted by the rod on the axis when AB is horizontal. ***E***

Solution:

a



The centre of mass, G, of the combined rod of mass m and particle of mass m will be half way between the middle of the rod and the particle at B, so $AG = \frac{3}{2}a$.

The moment of inertia of the combined rod and particle about A is given by

$$I = \frac{4}{3}ma^2 + m(2a)^2 = \frac{16}{3}ma^2$$

Let the angular speed of the system when AB is horizontal be ω.

Conservation of energy

$$\frac{1}{2}I\omega_0^2 - \frac{1}{2}I\omega^2 = 2mg \times \frac{3}{2}a$$

As the rod moves from the vertical to the horizontal the centre of mass of the system rises a distance $\frac{3}{2}a$.

$$\frac{1}{2} \times \frac{16}{3}ma^2 \times \left(\frac{9g}{4a}\right) - \frac{1}{2} \times \frac{16}{3}ma^2\omega^2 = 3mga$$

$$\frac{8}{3}ma^2\omega^2 = 6mga - 3mga$$

$$\omega^2 = \frac{9g}{8a} = \frac{9}{16} \times \frac{2g}{a}$$

$$\omega = \frac{3}{4}\sqrt{\left(\frac{2g}{a}\right)}, \text{ as required}$$

b When AB is horizontal, let the horizontal component of the force exerted by the rod on the axis be X

When AB is horizontal

$$\text{R}(\leftarrow) \quad X = (2m)r\dot{\theta}^2$$

$$= 2m\left(\frac{3}{2}a\right)\left(\frac{9g}{8a}\right) = \frac{27}{8}mg$$

The weight has no component in the horizontal direction.

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Review Exercise 2
Exercise A, Question 18

Question:

Particles P and Q have mass $3m$ and m respectively. Particle P is attached to one end of a light inextensible string and Q is attached to the other end. The string passes over a circular pulley which can freely rotate in a vertical plane about a fixed horizontal axis through its centre O. The pulley is modelled as a uniform circular disc of mass $2m$ and radius a. The pulley is sufficiently rough to prevent the string slipping. The system is at rest with the string taut. A third particle R of mass m falls freely under gravity from rest for a distance a before striking and adhering to Q. Immediately before R strikes Q, particles P and Q are at rest with the string taut.

a Show that, immediately after R strikes Q, the angular speed of the pulley is $\frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)}$.

When R strikes Q, there is an impulse in the string attached to Q.

b Find the magnitude of this impulse.

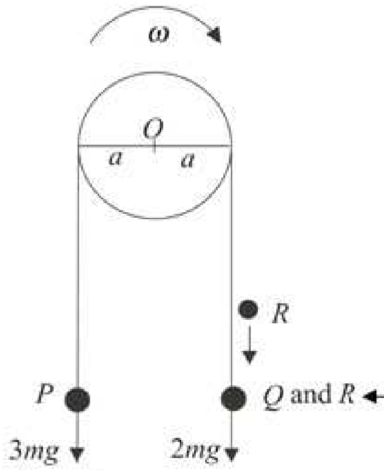
Given that P does not hit the pulley,

c find the distance that P moves upwards before first coming to instantaneous rest.

E

Solution:

a



R adheres to Q and, after the impact, they form a single particle of mass 2m.

$v^2 = u^2 + 2as$

$\quad = 0^2 + 2ga$

$v \ = \sqrt{(2ag)}$

You first need to find the speed of R immediately before it strikes Q.

Let the angular speed of the disc immediately after impact be ω.
Conservation of angular momentum about the centre of the pulley O

As the string does not slip, ω is also the angular speed of the whole system, consisting of the pulley, three particles and string.

$$m\sqrt{(2ga)} \times a = \left(3ma^2 + 2ma^2 + \frac{1}{2}(2m)a^2\right)\omega$$

$$= 6ma^2\omega$$

$$\omega = \frac{\sqrt{(2ga)}}{6a} = \frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)}, \text{ as required}$$

The moment of inertia of the system about O is given by
$I = I_P + I_{R\text{ and }Q} + I_{\text{pulley}}$
$\quad = (3m)a^2 + (2m)a^2 + \frac{1}{2}(2ma^2)$
$\quad = 6ma^2.$

b The impulse J in the string attached to Q is given by

$J = 2ma\omega - m\sqrt{(2ga)}$

$= 2ma \times \frac{1}{3}\sqrt{\left(\frac{g}{2a}\right)} - m\sqrt{(2ga)}$

$= \frac{1}{3}m\sqrt{(2ga)} - m\sqrt{(2ga)} = -\frac{2}{3}m\sqrt{(2ga)}$

Immediately before impact, only R is moving and has linear momentum $mv = m\sqrt{(2ga)}$.
Immediately after impact, Q and R are combined and have linear momentum $2mv$, where $v = a\omega$.

The magnitude of the impulse is $\frac{2}{3}m\sqrt{(2ga)}$.

c Let the distance that P moves upwards before first coming to instantaneous rest be s.
Conservation of energy
Kinetic energy lost = Potential energy gained

$$\frac{1}{2}I\omega^2 = 3mgs - 2mgs$$

$$\frac{1}{2}(6ma^2)\frac{g}{18a} = mgs$$

$$s = \frac{1}{6}a$$

P has risen a distance of s and gained potential energy $3mgs$. Q and R have fallen a distance s and have lost potential energy $2mgs$. The net gain in potential energy is mgs.
All of the kinetic energy of the system has been lost.

Using the expression for the angular speed found in part **a**.

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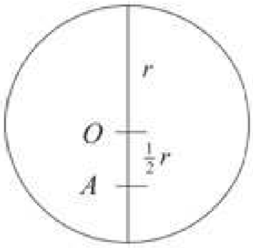
Review Exercise 2
Exercise A, Question 19

Question:

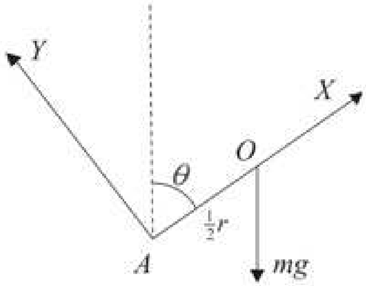
A uniform circular disc of mass m and radius r is free to rotate about a fixed smooth horizontal axis perpendicular to the plane of the disc and at a distance $\frac{1}{2}r$ from the centre of the disc. The disc is held at rest with the centre of the disc vertically above the axis.
Given that the disc is slightly disturbed from its position of rest, find the magnitude of the force on the axis when the centre of the disc is in the horizontal plane of the axis.
E

Solution:



Let O be the centre of the disc and the point where the axis of rotation meets the disc be A.
By the parallel axes theorem, the moment of inertia, I_A, of the disc about A is given by

$$I_A = I_O + m\left(\frac{1}{2}r\right)^2$$
$$= \frac{1}{2}mr^2 + \frac{1}{4}mr^2 = \frac{3}{4}mr^2$$



When AO has rotated through an angle θ, let the components of the force parallel and perpendicular to AO be X and Y respectively.
Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = mg\left(\frac{1}{2}r - \frac{1}{2}r\cos\theta\right)$$
$$\frac{1}{2}\times\frac{3}{4}mr^2\dot{\theta}^2 = \frac{1}{2}mgr(1-\cos\theta)$$
$$r\dot{\theta}^2 = \frac{4}{3}g(1-\cos\theta) \qquad ①$$

The centre of mass O of the disc has fallen a distance $\frac{1}{2}r - \frac{1}{2}r\cos\theta$.

R(‖ AO) $\mathbf{F} = m\mathbf{a}$

$$mg\cos\theta - X = m\left(\frac{1}{2}r\right)\dot{\theta}^2$$
$$X = mg\cos\theta - \frac{1}{2}m\times\frac{4}{3}g(1-\cos\theta)$$
$$= \frac{5}{3}mg\cos\theta - \frac{2}{3}mg$$

When AO is horizontal, $\theta = \frac{\pi}{2}$

$X = -\frac{2}{3}mg$

The component of the force parallel to AO is in the opposite direction to that indicated in the diagram. When drawing diagrams of the forces acting on the axis of rotation, you need not worry about the directions of the components. If you have them the wrong way round, this comes out in the working.

R($\perp AO$) $\mathbf{F} = m\mathbf{a}$

$$mg\sin\theta - Y = m\left(\frac{1}{2}r\right)\ddot{\theta}$$

$$Y = mg\sin\theta - \frac{1}{2}mr\ddot{\theta} \quad ②$$

Differentiating ① with respect to t

$$2r\dot{\theta}\ddot{\theta} = \frac{4}{3}g\sin\theta\,\dot{\theta}$$

$$r\ddot{\theta} = \frac{2}{3}g\sin\theta \quad ③$$

Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta}^2) = \frac{\mathrm{d}}{\mathrm{d}\dot{\theta}}(\dot{\theta}^2)\times\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 2\dot{\theta}\ddot{\theta} \text{ and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\times\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sin\theta\,\dot{\theta}$$

.

Substituting ③ into ②

$$Y = mg\sin\theta - \frac{1}{3}mg\sin\theta = \frac{2}{3}mg\sin\theta$$

When AO is horizontal, $\theta = \frac{\pi}{2}$

$$Y = \frac{2}{3}mg$$

Let the magnitude of the force on the axis be R

$$R^2 = X^2 + Y^2 = \frac{4}{9}m^2g^2 + \frac{4}{9}m^2g^2 = \frac{8}{9}m^2g^2$$

$$R = \frac{2\sqrt{2}}{3}mg$$

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Review Exercise 2
Exercise A, Question 20

Question:

A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. The rod is hanging in equilibrium with B below A when it is hit by a particle of mass m moving horizontally with speed v in a vertical plane perpendicular to the axis. The particle strikes the rod at B and immediately adheres to it.

a Show that the angular speed of the rod immediately after the impact is $\frac{3v}{8a}$.

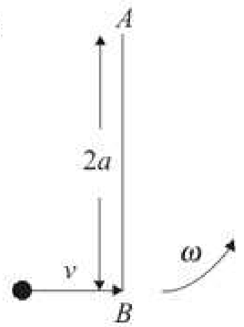
Given that the rod rotates through 120° before first coming to instantaneous rest,

b find v in terms of a and g,

c find, in terms of m and g, the magnitude of the vertical component of the force acting on the rod at A immediately after the impact. ***E***

Solution:

a



Let the angular speed of the rod immediately after the impact be ω.
The moment of inertia, I, of the rod combined with the particle about A is given by

$$I = I_{\text{rod}} + I_{\text{particle}}$$
$$= \frac{4}{3}ma^2 + m(2a)^2 = \frac{16}{3}ma^2$$

Conservation of angular momentum about A

$$mv \times 2a = \frac{16}{3}ma^2\omega$$
$$\omega = \frac{6mva}{16ma^2} = \frac{3v}{8a}\text{, as required}$$

Before the impact, the angular momentum of the particle about A is its linear momentum (mv) multiplied by the distance AB ($2a$).

b Let the centre of mass of the particle combined with the rod be G, then

$$AG = \frac{3}{2}a$$

The centre of mass, G, of the combined rod of mass m and particle of mass m will be half way between the middle of the rod and the particle at B, so $AG = \frac{3}{2}a$.

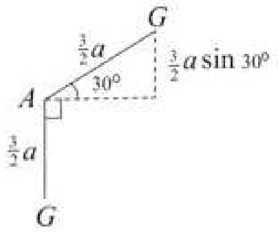
Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg\times\left(\frac{3}{2}a+\frac{3}{2}a\sin 30^\circ\right)$$

$$\frac{8}{3}ma^2\left(\frac{3v}{8a}\right)^2 = \frac{9}{2}mga$$

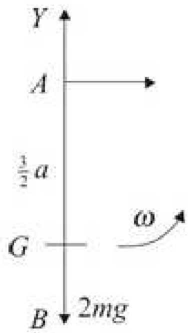
$$v^2 = \frac{9}{2}\times\frac{8}{3}ga = 12ga$$

$$v = \sqrt{(12ga)} = 2\sqrt{(3ga)}$$



As AB rotates through 120°, G rises a distance of $\frac{3}{2}a+\frac{3}{2}a\sin 30^\circ = \frac{9}{4}a$.

c



Let the magnitude of the vertical component of the force acting on the rod at A immediately after the impact be Y.

Immediately after the impact

R(↑) $\mathbf{F} = m\mathbf{a}$

The radial component of the acceleration is $r\dot{\theta}^2$.

$$Y - 2mg = 2mr\dot{\theta}^2$$

$$= 2m\left(\frac{3}{2}a\right)\left(\frac{3v}{8a}\right)^2 = \frac{27m}{64a}v^2$$

Using the answer to part **a**.

$$= \frac{27m}{64a}\times 12ga = \frac{81}{16}mg$$

Using the answer to part **b**.

$$Y = 2mg + \frac{81}{16}mg = \frac{113}{16}mg$$

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Review Exercise 2
Exercise A, Question 21

Question:

A uniform lamina, of mass m, has the form of a quadrant of a circle radius a.

a Show, by integration, that the moment of inertia of the lamina about an axis l perpendicular to the plane of the lamina and through the centre of the circle of which it is part, is $\frac{1}{2}ma^2$.

The lamina is free to rotate about l, which is horizontal, and when the centre of mass of the lamina is immediately below the axis of rotation the angular speed is Ω.

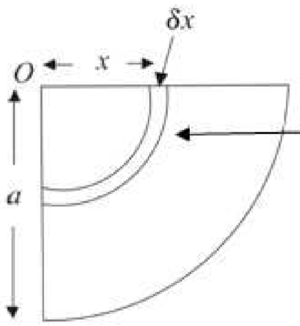
b Determine whether the lamina makes complete revolutions in the cases

i $\Omega = 2\sqrt{\left(\frac{g}{a}\right)}$,

ii $\Omega = 3\sqrt{\left(\frac{g}{a}\right)}$. ***E***

Solution:

a



You consider the quadrant to be made up of thin 'quarter rings', each of thickness δx.

The mass per unit area of the quadrant is $\frac{m}{\frac{1}{4}\pi a^2} = \frac{4m}{\pi a^2}$

The moment of inertia of an element of radius x and thickness δx is given by

$$\delta I = (\delta m)x^2 = \left(\frac{\pi x}{2} \times \frac{4m}{\pi a^2}\delta x\right)x^2 = \frac{2mx^3}{a^2}\delta x$$

Each 'quarter ring' has length one quarter of the circumference of the corresponding complete circle, that is $\frac{1}{4} \times 2\pi x = \frac{\pi x}{2}$.

For the whole quadrant

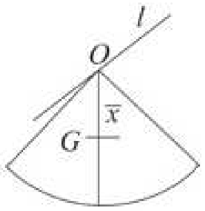
$$I = \sum \delta I = \sum \frac{2mx^3}{a^2}\delta x$$

As $\delta x \to 0$

$$I = \int_0^a \frac{2mx^3}{a^2}\mathrm{d}x = \left[\frac{2mx^4}{4a^2}\right]_0^a$$

$$= \frac{2ma^4}{4a^2} = \frac{1}{2}ma^2, \text{ as required}$$

b



Let G be the centre of mass of the quadrant, O the centre of the circle of which the quadrant is part, and $OG = \bar{x}$.

$$\bar{x} = \frac{2a \sin \alpha}{3\alpha}$$

This formula for the centre of mass of a sector of a circle, radius r, angle at centre 2α, is one of the formulae for module M2 given in the Formulae Booklet. For module M5 you are expected to know the specifications for modules M1, M2, M3 and M4 together with their associated formulae.

With $\alpha = \frac{\pi}{4}$

$$\bar{x} = \frac{2a \sin \frac{\pi}{4}}{3 \times \frac{\pi}{4}} = \frac{8a \times \frac{1}{\sqrt{2}}}{3\pi} = \frac{4\sqrt{2}a}{3\pi}$$

For complete revolutions, by energy,

$$\frac{1}{2} I\Omega^2 \geq mg \times 2\bar{x}$$

For complete revolutions, the kinetic energy at the lowest point must be sufficient to allow the centre of mass of the lamina, G, to rise from the point where G is vertically below O to the point where G is vertically above O; that is a distance of $2\bar{x}$.

$$\frac{1}{4} ma^2\Omega^2 \geq mg \frac{8\sqrt{2}a}{3\pi}$$

$$\Omega^2 \geq \frac{32\sqrt{2}g}{3\pi a}$$

$$\Omega \geq \left(\frac{32\sqrt{2}}{3\pi}\right)^{\frac{1}{2}} \sqrt{\left(\frac{g}{a}\right)} \approx 2.19 \sqrt{\left(\frac{g}{a}\right)}$$

i If $\Omega = 2\sqrt{\left(\frac{g}{a}\right)}$, as $2 < 2.19$, the lamina does not make complete revolutions.

ii If $\Omega = 3\sqrt{\left(\frac{g}{a}\right)}$, as $3 > 2.19$, the lamina does make complete revolutions.

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Review Exercise 2
Exercise A, Question 22

Question:

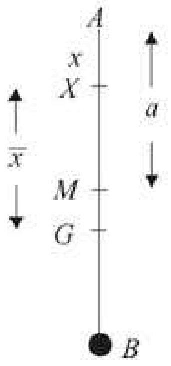
A uniform rod AB, of length $2a$ and mass $6m$, has a particle of mass $2m$ attached at B. The rod is free to rotate in a vertical plane about a smooth fixed vertical axis perpendicular to the rod and passing through a point X of the rod so that $AX = x$, where $x < a$.

a Show that the moment of inertia of the system about this axis is $4m(4a^2 - 5ax + 2x^2)$.

b Find the period of small oscillations of the system about its equilibrium position with B below A. ***E***

Solution:

a



Let the mid-point of the rod be M.
The moment of inertia of the rod, I_R, about X is given by

$$I_R = \frac{1}{3}(6m)a^2 + 6m \times XM^2$$ ← Using the parallel axes theorem.

$$= 2ma^2 + 6m(a-x)^2$$

The moment of inertia, I, of the rod and particle combined is given by

$$I = I_R + I_P$$
$$= 2ma^2 + 6m(a-x)^2 + 2m(2a-x)^2$$
$$= 2ma^2 + 6ma^2 - 12max + 6mx^2 + 8ma^2 - 8max + 2mx^2$$
$$= 16ma^2 - 20max + 8mx^2$$
$$= 4m(4a^2 - 5ax + 2x^2), \text{as required}$$

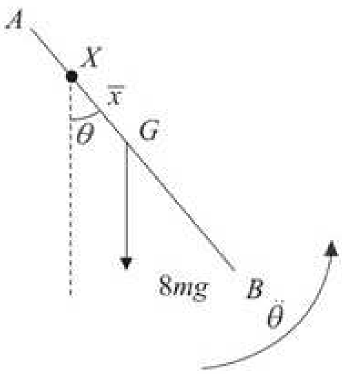
b Let the centre of mass of the rod and particle combined be G and $GX = \bar{x}$.

$$\text{M}(X)\ 8mg\bar{x} = 6mg(a-x) + 2mg(2a-x)$$

$$8\bar{x} = 6a - 6x + 4a - 2x = 10a - 8x$$

$$\bar{x} = \frac{5}{4}a - x$$

The total weight of the rod and particle acts at G and you can find the position of G by taking moments about X. You can then use this distance to write down the equation of rotational motion.



The equation of rotational motion about X is

$$L = I\ddot{\theta}$$

$$-8mg\bar{x}\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{8mg\bar{x}}{I}\sin\theta$$

The moment of the weight about the axis through X is tending to make θ decrease and so has a negative sign in the equation of rotational motion.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{8mg\bar{x}}{I}\theta = -\frac{8mg\left(\frac{5}{4}a - x\right)}{4m\left(4a^2 - 5ax + 2x^2\right)}\theta$$

$$= -\frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}.$$

So $\omega = \left(\frac{g(5a-4x)}{2\left(4a^2 - 5ax + 2x^2\right)}\right)^{\frac{1}{2}}$ and you use this expression in the formula for the period of simple harmonic motion $T = \frac{2\pi}{\omega}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{2\left(4a^2 - 5ax + 2x^2\right)}{g(5a-4x)}\right)^{\frac{1}{2}}$$

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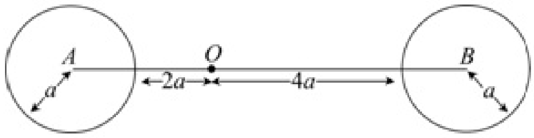
Review Exercise 2
Exercise A, Question 23

Question:



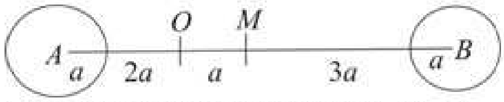
A body consists of two uniform circular discs, each of mass m and radius a, with a uniform rod. The centres of the discs are fixed to the ends A and B of the rod, which has mass $3m$ and length $8a$. The discs and the rod are coplanar, as shown in the figure. The body is free to rotate in a vertical plane about a smooth fixed horizontal axis. The axis is perpendicular to the plane of the discs and passes through the point O of the rod, where $AO = 3a$.

a Show that the moment of inertia of the body about the axis is $54ma^2$.
The body is held at rest with AB horizontal and is then released. When the body has turned through an angle of $30°$, the rod AB strikes a small fixed smooth peg P where $OP = 3a$. Given that the body rebounds from the peg with its angular speed halved by the impact,

b show that the magnitude of the impulse exerted on the body by the peg at the impact is $9m\sqrt{\left(\frac{5ga}{6}\right)}$. ***E***

Solution:

a



In this diagram, M is the mid-point of AB. By symmetry, M is the centre of mass of the body.

The moment of inertia of the rod AB, I_{AB}, about O is given by

$$I_{AB} = \frac{1}{3}(3m)(4a)^2 + (3m)a^2 = 19ma^2$$

The moment of inertia of the disc centre A, I_A, about O is given by

$$I_A = \frac{1}{2}ma^2 + m(3a)^2 = \frac{19}{2}ma^2$$

The moment of inertia of the disc centre B, I_B, about O is given by

$$I_B = \frac{1}{2}ma^2 + m(5a)^2 = \frac{51}{2}ma^2$$

You use the parallel axes theorem for all three of the component parts which make up the body.

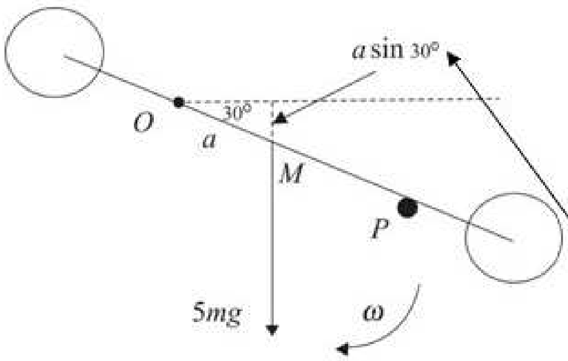
The moment of inertia of the whole body, I, about O is given by

$$I = I_{AB} + I_A + I_B$$

$$= 19ma^2 + \frac{19}{2}ma^2 + \frac{51}{2}ma^2$$

$= 54ma^2$, as required

b



As the body rotates through 30°, the centre of mass of the body, M, falls vertically a distance $a\sin 30° = \frac{1}{2}a$.

Let the angular speed of the body immediately before the impact be ω.
Conservation of energy
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\omega^2 = 5mg\times\frac{1}{2}a$$

$$\omega^2 = \frac{5mga}{I} = \frac{5g}{54a}$$

$$\omega = \sqrt{\left(\frac{5g}{54a}\right)} = \frac{1}{3}\sqrt{\left(\frac{5g}{6a}\right)}$$

Let the angular speed of the body immediately after the impact be ω'.

$$\omega' = \frac{1}{6}\sqrt{\left(\frac{5g}{6a}\right)}$$

The question gives you that the angular speed is halved by the impact.

Let J be the magnitude of the impulse exerted on the body by the peg at the impact.

Moment of impulse = change in angular momentum

$$-3aJ = I(-\omega') - I\omega$$

The impulse is in a direction which decreases the angle the rod makes with the horizontal.

$$3aJ = I(\omega + \omega')$$

$$= 54ma^2\left[\frac{1}{3}\sqrt{\left(\frac{5g}{6a}\right)} + \frac{1}{6}\sqrt{\left(\frac{5g}{6a}\right)}\right]$$

The angular velocities before and after impact are in opposite senses. As drawn, ω is clockwise and ω' is anti-clockwise.

$$= 54ma^2\times\frac{1}{2}\sqrt{\left(\frac{5g}{6a}\right)}$$

$$J = \frac{27ma^2}{3a}\sqrt{\left(\frac{5g}{6a}\right)} = 9m\sqrt{\left(\frac{5ga}{6}\right)}, \text{ as required}$$

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Review Exercise 2
Exercise A, Question 24

Question:

a Show that the moment of inertia of a uniform solid right circular cone, of mass m, height h and base radius a, about a line through its vertex and perpendicular to its axis of symmetry is

$$\frac{3}{20}m(a^2+4h^2)$$

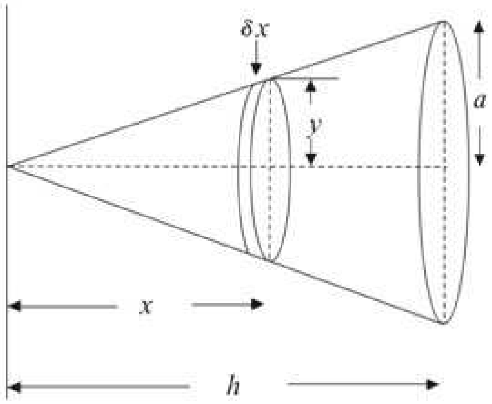
[You may assume that the moment of inertia of a uniform circular disc, of mass M and radius R, about a diameter is $\frac{1}{4}MR^2$.]

A cone, with $h=\frac{2}{3}a$, is free to rotate about a smooth horizontal axis through its vertex.

b Find the period of small oscillations under gravity about the stable position of equilibrium. ***E***

Solution:

a Axis of rotation



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{m}{\frac{1}{3}\pi a^2 h} = \frac{3m}{\pi a^2 h}$$

The moment of inertia of an elementary disc about the axis of rotation is given by

$$\delta I = \frac{1}{4}(\delta m)y^2 + (\delta m)x^2$$

The question specifies that you can use the formula $I = \frac{1}{4}MR^2$ for the thin disc. You use this, with $M = \delta m$ and $R = y$, and the parallel axes theorem to form an expression for the moment of inertia, δI, of the thin disc about the axis of rotation.

By similar triangles

$$\frac{y}{x} = \frac{a}{h} \Rightarrow y = \frac{ax}{h}$$

Hence

$$\delta I = \frac{1}{4}(\pi y^2 \delta x)\left(\frac{3m}{\pi a^2 h}\right)y^2 + (\pi y^2 \delta x)\left(\frac{3m}{\pi a^2 h}\right)x^2$$

$$= \frac{3m}{a^2 h}\left(\frac{y^4}{4} + y^2 x^2\right)\delta x = \frac{3m}{a^2 h}\left(\frac{a^4 x^4}{4h^4} + \frac{a^2 x^4}{h^2}\right)\delta x$$

$$= \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit length $\frac{3m}{\pi a^2 h}$.

For the complete cone

$$I = \sum \delta I = \sum \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \delta x$$

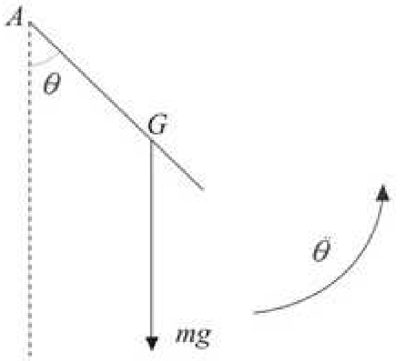
As $\delta x \to 0$

$$I = \int_0^h \frac{3m}{4h^5}(a^2 + 4h^2)x^4 \, \mathrm{d}x = \frac{3m}{4h^5}(a^2 + 4h^2)\left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{3m}{4h^5}(a^2 + 4h^2)\left(\frac{h^5}{5} - 0\right) = \frac{3}{20}m(a^2 + 4h^2), \text{ as required}$$

As the thin, or elementary, discs range from the vertex to the base, x ranges from 0 to h. So 0 and h are the limits of integration.

b



Let A be the vertex of the cone and G the centre of mass of the cone.

$AG = \frac{3}{4}h = \frac{3}{4} \times \frac{2}{3}a = \frac{1}{2}a$

A formula for the centre of mass of a cone is given among the formulae for module M3 in the Formulae Booklet. For module M5 you are expected to know the specifications for modules M1, M2, M3 and M4 together with their associated formulae.

As $h = \frac{2}{3}a$,

$I = \frac{3}{20}m\left(a^2 + 4h^2\right) = \frac{3}{20}m\left(a^2 + 4 \times \frac{4a^2}{9}\right)$

$= \frac{3}{20}m \times \frac{25}{9}a^2 = \frac{5}{12}ma^2$

The equation of motion about A is

$L = I\ddot{\theta}$

$-mg\left(\frac{a}{2}\right)\sin\theta = \frac{5}{12}ma^2\ddot{\theta}$

$\ddot{\theta} = -\frac{6g\sin\theta}{5a}$

The moment of the weight about A is tending to make θ decrease and so has a negative sign in the equation of rotational motion.

For small $\theta, \sin\theta \approx \theta$

Hence

$\ddot{\theta} = -\frac{6g}{5a}\theta$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{6g}{5a}$.

The period of small oscillations is given by

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{6g}{5a}}} = 2\pi\sqrt{\frac{5a}{6g}}$

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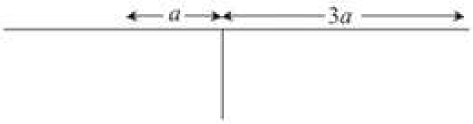
Review Exercise 2
Exercise A, Question 25

Question:



A rough uniform rod, of mass m and length $4a$, is held on a rough horizontal table. The rod is perpendicular to the edge of the table and a length $3a$ projects horizontally over the edge, as shown in the figure.

a Show that the moment of inertia of the rod about the edge of the table is $\frac{7}{3}ma^2$.

The rod is released from rest and rotates about the edge of the table. When the rod has turned through an angle θ, its angular speed is $\dot{\theta}$. Assuming that the rod has not started to slip,

b show that $\dot{\theta}^2 = \frac{6g\sin\theta}{7a}$,

c find the angular acceleration of the rod,

d find the normal reaction of the table on the rod.

The coefficient of friction between the rod and the edge of the table is μ.

e Show that the rod starts to slip when $\tan\theta = \frac{4}{13}\mu$. ***E***

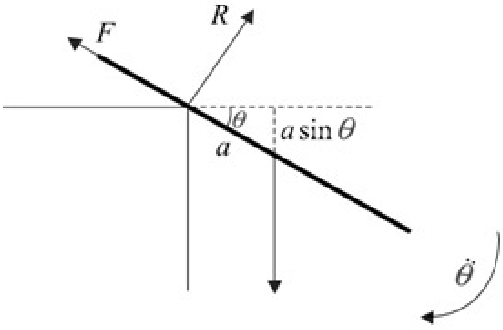
Solution:

a Using the parallel axes theorem, the moment of inertia, I, of the rod about the edge of the table is given by

$I = \frac{1}{3}m(2a)^2 + ma^2 = \frac{7}{3}ma^2$, as required

Using the standard formula for the moment of inertia of a rod about its centre $\frac{1}{3}ml^2$ with $l = 2a$. The centre of mass of the rod is the distance a from the edge of the table.

b



Conservation of energy

Kinetic energy gained = Potential energy lost

$$\frac{1}{2}I\dot{\theta}^2 = mga\sin\theta$$

$$\frac{7}{6}ma^2\dot{\theta}^2 = mga\sin\theta$$

$$\dot{\theta}^2 = \frac{6g\sin\theta}{7a}, \text{ as required}$$

As the rod rotates through θ, its centre of mass falls a vertical distance $a\sin\theta$.

c Differentiate the result of **b** implicitly throughout with respect to t

$$2\dot{\theta}\ddot{\theta} = \frac{6g\cos\theta}{7a}\dot{\theta}$$

$$\ddot{\theta} = \frac{3g\cos\theta}{7a}$$

Using the chain rule,

$\frac{d}{dt}(\dot{\theta}^2) = \frac{d}{d\dot{\theta}}(\dot{\theta}^2) \times \frac{d\dot{\theta}}{dt} = 2\dot{\theta}\ddot{\theta}$ and

$\frac{d}{dt}(\sin\theta) = \frac{d}{d\theta}(\sin\theta) \times \frac{d\theta}{dt} = \cos\theta\,\dot{\theta}$

.

d Let the reaction on the rod, normal to the rod, at the edge of the table be R.

R($\perp AB$) $\mathbf{F} = m\mathbf{a}$

$mg\cos\theta - R = ma\ddot{\theta}$

The component of the weight is in the direction of θ increasing and R is in the direction of θ decreasing.

$$R = mg\cos\theta - ma\ddot{\theta} = mg\cos\theta - ma\frac{3g}{7a}\cos\theta$$

Using the result of part **c**.

$$= \frac{4mg\cos\theta}{7}$$

e Let F be the frictional force at the edge of the table

R($\parallel AB$) $\mathbf{F} = m\mathbf{a}$

$F - mg\sin\theta = ma\dot{\theta}^2$

The radial component of the acceleration is $r\dot{\theta}^2$.

$$F = mg\sin\theta + ma\dot{\theta}^2 = mg\sin\theta + ma\frac{6g\sin\theta}{7a}$$

$$= \frac{13mg\sin\theta}{7}$$

Using the result of part **b**.

As the rod starts to slip

$F = \mu R$

$$\frac{13mg\sin\theta}{7} = \mu\frac{4mg\cos\theta}{7}$$

$\tan\theta = \frac{4}{13}\mu$, as required

As $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

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Review Exercise 2
Exercise A, Question 26

Question:

a Show, by integration, that the moment of inertia of a uniform circular disc, of mass M and radius a, about an axis which passes through its centre and is perpendicular to its plane is $\frac{1}{2}Ma^2$.

b Without further integration, deduce the moment of inertia of the disc

i about an axis perpendicular to its plane and passing through a point on its circumference,

ii about a diameter.

A uniform disc, of mass M and radius a, is suspended from a smooth pivot on its circumference and rests in equilibrium.

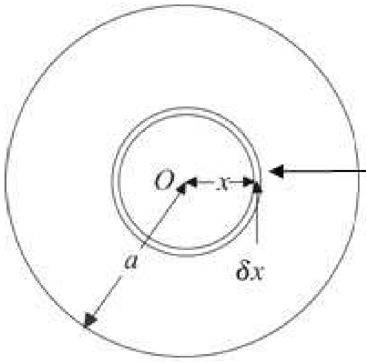
c Calculate the period of small oscillations when the centre of the disc is slightly displaced

i in the plane of the disc,

ii perpendicular to the plane of the disc. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x \delta x$.

Let the centre of the disc be O.

The mass per unit area of the disc is $\dfrac{M}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x \delta x \times \frac{M}{\pi a^2}\right)x^2 = \frac{2M}{a^2}x^3 \delta x$$

The moment of inertia of the disc, I, is given by

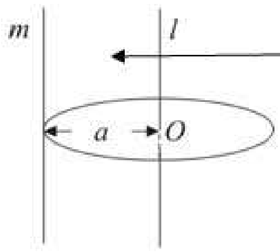
$$I = \sum \delta I = \sum \frac{2M}{a^2}x^3 \delta x$$

As $\delta x \to 0$

$$I = \int_0^a \frac{2M}{a^2}x^3 \, \mathrm{d}x = \frac{2M}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2M}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$$= \frac{1}{2}Ma^2, \text{ as required}$$

b i



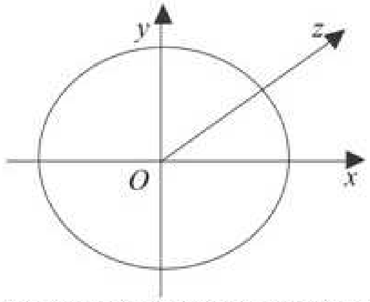
In part **a**, you have shown that the moment of inertia about the axis l, an axis through the centre perpendicular to the plane of the disc, is $\frac{1}{2}Ma^2$. You find the moment of inertia about the axis m, an axis parallel to l, through a point on the circumference of the disc, using the result of part **a** and the parallel axes theorem.

By the parallel axes theorem

$$I_m = I_l + Ma^2$$

$$= \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$$

ii



By the perpendicular axes theorem, the moment of inertia, I_{Ox}, about a diameter Ox through O is given by

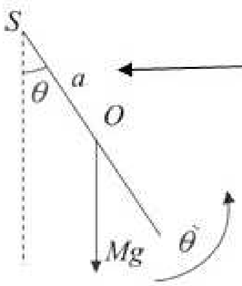
$$I_{Ox} + I_{Oy} = I_{Oz}$$

$$2I_{Ox} = \frac{1}{2}Ma^2$$

$$I_{Ox} = \frac{1}{4}Ma^2$$

By symmetry the moment of inertia about the axis Ox equals the moment of inertia about the axis Oy.

c



In this diagram, the smooth pivot is S and the angle SO makes with the downward vertical is θ.

The equation of rotational motion about S is

$$L = I\ddot{\theta}$$

$$-Mga\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{Mga}{I}\sin\theta$$

By leaving the moment of inertia as I, you can find the equations of angular motion for both parts **c i** and **c ii** together.

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{Mga}{I}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$,
the motion is approximately simply harmonic, with

$$\omega^2 = \frac{Mga}{I}.$$

i $I = \frac{3}{2}Ma^2$

The axis of rotation is the same as in part **b i**.

$$\omega^2 = \frac{Mga}{I} = \frac{Mga}{\frac{3}{2}Ma^2} = \frac{2g}{3a}$$

Hence the period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{3a}{2g}\right)} = \pi\sqrt{\left(\frac{6a}{g}\right)}$$

ii By the parallel axes theorem the moment of inertia, I, about a tangent to the disc is given by

$$I = \frac{1}{4}Ma^2 + Ma^2 = \frac{5}{4}Ma^2$$

The axis of rotation in this part is a tangent to the disc which is parallel to a diameter and, so, you use the result of part **b ii** together with the parallel axes theorem.

$$\omega^2 = \frac{Mga}{I} = \frac{Mga}{\frac{5}{4}Ma^2} = \frac{4g}{5a}$$

Hence the period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{5a}{4g}\right)} = \pi\sqrt{\left(\frac{5a}{g}\right)}$$

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Review Exercise 2
Exercise A, Question 27

Question:

A uniform plane circular disc, of mass m and radius a, hangs in equilibrium from a point B on its circumference. The disc is free to rotate about a fixed smooth horizontal axis which is in the plane of the disc and tangential to the disc at B. A particle P, of mass m, is moving horizontally with speed u in a direction which is perpendicular to the plane of the disc. At time $t = 0, P$ strikes the disc at its centre and adheres to the disc.

a Show that the angular speed of the disc immediately after it has been struck by P is $\frac{4u}{9a}$.

It is given that $u^2 = \frac{1}{10}ag$, and that air resistance is negligible.

b Find the angle through which the disc turns before it first comes to instantaneous rest.

The disc first returns to its initial position at time $t = T$.

c **i** Write down an equation of motion for the disc.

ii Hence find T in terms of a, g and m, using a suitable approximation which should be justified. ***E***

Solution:

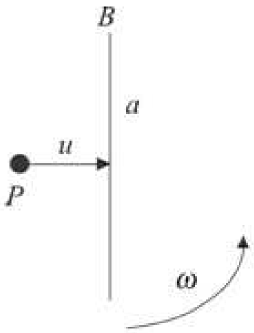
a The moment of inertia of the disc about a diameter is $\frac{1}{4}ma^2$.

> This formula is given the Formulae Booklet and, if you are not asked to prove it, it can be quoted.

The moment of inertia, I, of the disc about a tangent is given, using the parallel axes theorem, by

$$I = \frac{1}{4}ma^2 + ma^2 = \frac{5}{4}ma^2$$

> A tangent is parallel to a diameter and the distance between the tangent and the diameter is the radius a.



> This diagram is drawn looking at the disc edge on. ω is the angular speed of the disc and particle immediately after impact.

The moment of inertia, I', of the disc and P about the axis through B is given by

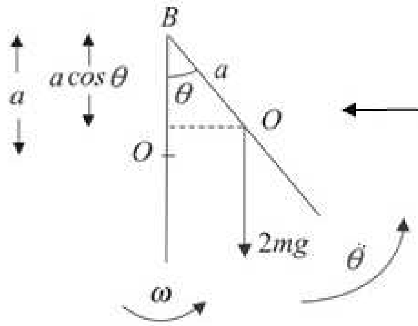
$$I' = \frac{5}{4}ma^2 + ma^2 = \frac{9}{4}ma^2$$

Conservation of angular momentum about B

$$mu \times a = I'\omega = \frac{9}{4}ma^2\omega$$

$$\omega = \frac{4u}{9a}, \text{ as required}$$

b



This diagram is drawn looking at the disc edge on. O is the centre of the disc.

As $u^2 = \frac{1}{10}ag$,

$$\omega^2 = \frac{16u^2}{81a^2} = \frac{16}{81a^2} \times \frac{1}{10}ag = \frac{8g}{405a}$$

Using the answer to part **a**.

Let the disc first come to rest when $\theta = \alpha$

Conservation of energy

$$\frac{1}{2}I\omega^2 = 2mg(a - a\cos\alpha)$$

From the diagram, as the disc swings through θ, the centre of the disc rises a vertical distance of $a - a\cos\theta$.

$$\frac{1}{2} \times \frac{9ma^2}{4} \times \frac{8g}{405a} = 2mga(1 - \cos\alpha)$$

$$\frac{1}{45} = 2(1 - \cos\alpha) \Rightarrow \cos\alpha = \frac{89}{90}$$

$$\alpha = \arccos\left(\frac{89}{90}\right) = 8.5^\circ, \text{ to the nearest } 0.1^\circ.$$

c i The equation of rotational motion about B is

$$L = I\ddot{\theta}$$

$$-2mga\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{2mga}{I}\sin\theta = -\frac{2mga}{\frac{9}{4}ma^2}\sin\theta = -\frac{8g}{9a}\sin\theta$$

ii For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{8g}{9a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{8g}{9a}.$$

Hence

$$T = \frac{1}{2} \times \frac{2\pi}{\omega} = \pi\sqrt{\left(\frac{9a}{8g}\right)} = \frac{3\pi}{2}\sqrt{\left(\frac{a}{2g}\right)}$$

The time the disc takes to moves from the centre of its motion to its amplitude and then back to the centre of its motion is one half of a complete period.

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Review Exercise 2
Exercise A, Question 28

Question:

Four uniform rods, each of mass m and length $2a$, are joined together at their ends to form a plane rigid square framework $ABCD$ of side $2a$. The framework is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. The axis is perpendicular to the plane of the framework.

a Show that the moment of inertia of the framework about the axis is $\frac{40ma^2}{3}$.

The framework is slightly disturbed from rest when C is vertically above A. Find

b the angular acceleration of the framework when AC is horizontal,

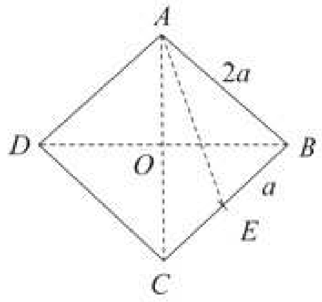
c the angular speed of the framework when AC is horizontal,

d the magnitude of the force acting on the framework at A when AC is horizontal.

E

Solution:

a



Let E be the mid-point of BC.

$AE^2 = a^2 + (2a)^2 = 5a^2$ ← Using Pythagoras' Theorem.

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{3}ma^2 + mAE^2$$

$$= \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$$

The moment of inertia of the framework about the axis through A is given by

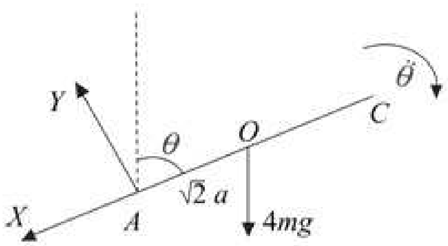
$$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$$

By symmetry, the moment of inertia of the rod CD about the axis through A will be the same as the moment of inertia of the rod BC.

$$= \frac{4}{3}ma^2 + \frac{16}{3}ma^2 + \frac{16}{3}ma^2 + \frac{4}{3}ma^2$$

$$= \frac{40}{3}ma^2, \text{ as required}$$

b Let O be the centre of mass of the lamina



$AO^2 + BO^2 = AB^2 = (2a)^2 = 4a^2$ ← By symmetry, $AO = BO$.

$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2 \Rightarrow AO = \sqrt{2}a$

Equation of angular motion about A

$$L = I\ddot{\theta}$$

$$4mg\sqrt{2}a\sin\theta = \frac{40}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{3g\sqrt{2}}{10a}\sin\theta$$

When $\theta = \frac{\pi}{2}$

$$\ddot{\theta} = \frac{3g\sqrt{2}}{10a}$$

When AC is horizontal,
$\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$.

c Conservation of energy

$$\frac{1}{2}I\dot{\theta}^2 = 4mg(\sqrt{2}a - \sqrt{2}a\cos\theta)$$

Initially the framework has no kinetic energy. As the framework rotates through θ, O falls a vertical distance $\sqrt{2}a - \sqrt{2}a\cos\theta$.

$$\frac{1}{2}\times\frac{40}{3}ma^2\dot{\theta}^2 = 4\sqrt{2}mga(1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{3g\sqrt{2}}{5a}(1-\cos\theta)$$

When $\theta = \frac{\pi}{2}$

$$\dot{\theta}^2 = \frac{3g\sqrt{2}}{5a}$$

$$\dot{\theta} = \left(\frac{3g\sqrt{2}}{5a}\right)^{\frac{1}{2}}$$

d Let the magnitude of the force acting on the framework at A be R and the components of this force parallel and perpendicular to AO be X and Y respectively.

R($\parallel AO$)

$$X + 4mg\cos\theta = 4m(\sqrt{2}a)\dot{\theta}^2$$

When $\theta = \frac{\pi}{2}$

$$X + 4\sqrt{2}ma \times \frac{3g\sqrt{2}}{5a} = \frac{24}{5}mg$$

Using the result of part **c** and $\cos\frac{\pi}{2} = 0$.

R($\perp AO$)

$$4mg\sin\theta - Y = 4m(\sqrt{2}a)\ddot{\theta}$$

When $\theta = \frac{\pi}{2}$

$$Y = 4mg - 4\sqrt{2}ma \times \frac{3g\sqrt{2}}{10a}$$

Using the result of part **b** and $\sin\frac{\pi}{2} = 1$.

$$= 4mg - \frac{12}{5}mg = \frac{8}{5}mg$$

$$R^2 = X^2 + Y^2$$

$$= \left(\frac{24}{5}mg\right)^2 + \left(\frac{8}{5}mg\right)^2 = \frac{640}{25}m^2g^2$$

$$R = \frac{8\sqrt{10}}{5}mg$$

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Review Exercise 2
Exercise A, Question 29

Question:

a Prove, using integration, that the moment of inertia of a uniform circular disc, of mass m and radius a, about an axis through its centre O perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

The line AB is a diameter of the disc and P is the mid-point of OA. The disc is free to rotate about a fixed smooth horizontal axis L. The axis lies in the plane of the disc, passes through P and is perpendicular to OA. A particle of mass m is attached to the disc at A and a particle of mass $2m$ is attached to the disc at B.

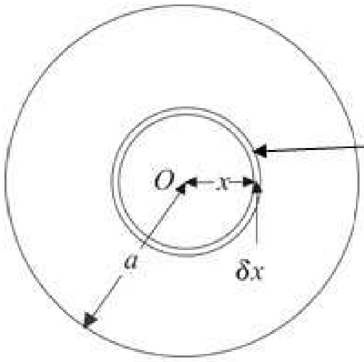
b Show that the moment of inertia of the loaded disc about L is $\frac{21}{4}ma^2$.

At time $t = 0, PB$ makes a small angle with the downward vertical through P and the loaded disc is released from rest. By obtaining an equation of motion for the disc and using a suitable approximation,

c find the time when the loaded disc first comes to instantaneous rest. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x\delta x$.

The mass per unit area of the disc is $\frac{m}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x\delta x \times \frac{m}{\pi a^2}\right)x^2 = \frac{2m}{a^2}x^3\delta x$$

The moment of inertia of the disc, I, is given by

$$I = \sum \delta I = \sum \frac{2m}{a^2}x^3\delta x$$

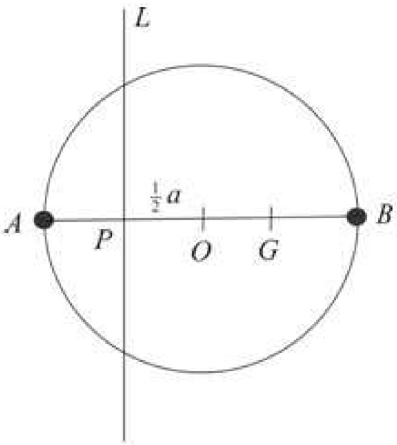
As $\delta x \to 0$

$$I = \int_0^a \frac{2m}{a^2}x^3 \,\mathrm{d}x = \frac{2m}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2m}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$$= \frac{1}{2}ma^2, \text{ as required}$$

This is a standard result which you should be able to prove. You are expected to be able to prove all of the standard results given in the Formulae Booklet and you should practise writing these out.

b



The moment of inertia of the disc about L is given by

$$I_{\text{disc}} = \frac{1}{4}ma^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{2}ma^2$$

Using the standard result for the moment of inertia of a disc about a diameter and the parallel axes theorem.

The moment of inertia of the loaded disc, I, is given by

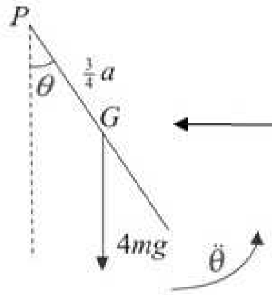
$$I = I_{\text{disc}} + I_A + I_B$$
$$= \frac{1}{2}ma^2 + m\left(\frac{1}{2}a\right)^2 + 2m\left(\frac{3}{2}a\right)^2$$
$$= \frac{1}{2}ma^2 + \frac{1}{4}ma^2 + \frac{9}{2}ma^2 = \frac{21}{4}ma^2, \text{ as required}$$

$AP = \frac{1}{2}a$ and $BP = \frac{3}{2}a$.

c Let G be the centre of mass of the loaded plate.
M(L)

$$4m \times PG = m \times \frac{1}{2}a + 2m \times \frac{3}{2}a - m \times \frac{1}{2}a$$
$$4mPG = 3ma$$
$$PG = \frac{3}{4}a$$

You take moments about L to locate the position of G. As A is on the other side of L from O and B, the moment of the particle of mass m has a negative sign in this equation.



The total weight, $4mg$, of the loaded plate acts at G.

The equation of rotational motion about P is

$$L = I\ddot{\theta}$$
$$-4mg \times \frac{3}{4}a\sin\theta = \frac{21}{4}ma^2\ddot{\theta}$$
$$\ddot{\theta} = -\frac{4g}{7a}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$
Hence

$$\ddot{\theta} = -\frac{4g}{7a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with

$$\omega^2 = \frac{4g}{7a}.$$

The time, after release, for the loaded disc to first come to instantaneous rest is given by

$$t = \frac{1}{2}T = \frac{\pi}{\omega} = \pi\sqrt{\left(\frac{7a}{4g}\right)} = \frac{\pi}{2}\sqrt{\left(\frac{7a}{g}\right)}$$

This time is the time for one half of a complete oscillation.

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Review Exercise 2
Exercise A, Question 30

Question:

A uniform lamina of mass m is in the shape of an equilateral triangle ABC of perpendicular height h. The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L through A and perpendicular to the plane of the lamina.

a Show, by integration, that the moment of inertia of the lamina about L is $\frac{5}{9}mh^2$.

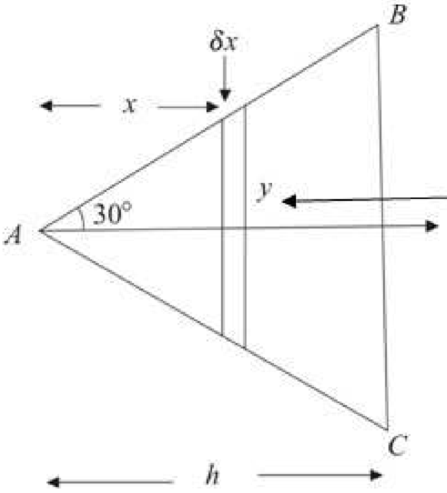
The centre of mass of the lamina is G. The lamina is in equilibrium, with G below A, when it is given an angular speed $\sqrt{\left(\frac{6g}{5h}\right)}$.

b Find the angle between AG and the downward vertical, when the lamina first comes to rest.

c Find the greatest magnitude of the angular acceleration during the motion. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. A rod is a distance x from A.

By trigonometry

$$\frac{\frac{1}{2}BC}{h} = \tan 30°$$

$$BC = 2h\tan 30° = \frac{2}{\sqrt{3}}h$$

To find an expression for the mass per unit area of the triangle, you have to first obtain an expression for the area of the triangle in terms of h.

The area of the triangle is given by

$$\frac{1}{2}BC \times h = \frac{1}{\sqrt{3}}h \times h = \frac{1}{\sqrt{3}}h^2$$

The mass per unit area of the triangle is

$$\frac{m}{\frac{1}{\sqrt{3}}h^2} = \frac{\sqrt{3}m}{h^2}$$

The moment of inertia, δI, of one elementary rod about L is given by

You use the standard formula for the moment of inertia of a rod about its centre and the parallel axes theorem to find the moment of inertia of a rod about axis L through A.

$$\delta I = \frac{1}{3}(\delta m)y^2 + (\delta m)x^2$$

$$= (\delta m)\left(\frac{1}{3}y^2 + x^2\right)$$

$$= \left(2y\delta x \times \frac{\sqrt{3}m}{h^2}\right)\left(\frac{1}{3}y^2 + x^2\right)$$

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By trigonometry

$$\frac{y}{x} = \tan 30° = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}x$$

Hence

$$\delta I = \left(2\frac{1}{\sqrt{3}}x\delta x \times \frac{\sqrt{3}m}{h^2}\right)\left(\frac{1}{3}\left(\frac{1}{\sqrt{3}}x\right)^2 + x^2\right)$$

$$= \frac{2mx}{h^2}\delta x \times \left(\frac{1}{9}x^2 + x^2\right) = \frac{20m}{9h^2}x^3\delta x$$

$$I = \sum \delta I = \sum \frac{20m}{9h^2} x^3 \delta x$$

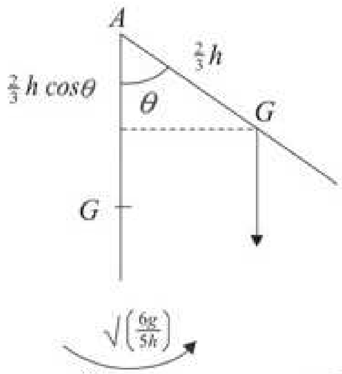
As $\delta x \to 0$

$$I = \frac{20m}{9h^2}\int_0^h x^3 \, dx = \frac{20m}{9h^2}\left[\frac{x^4}{4}\right]_0^h$$

$$= \frac{20m}{9h^2} \times \frac{h^4}{4} = \frac{5}{9}mh^2, \text{ as required}$$

As the rods range from A to BC, x ranges from 0 to h, so 0 and h are the limits of integration.

b



$$AG = \frac{2}{3}h$$

Using the standard result that the centre of mass of a triangle is $\frac{2}{3}$ along the median from the vertex.

Conservation of energy

$$\frac{1}{2}I\omega^2 = mg\left(\frac{2}{3}h - \frac{2}{3}h\cos\theta\right)$$

When the lamina comes to rest, all of the original kinetic energy has been converted to potential energy

$$\frac{5}{18}mh^2 \times \frac{6g}{5h} = \frac{2}{3}mgh(1-\cos\theta)$$

$$\frac{1}{3} = \frac{2}{3}(1-\cos\theta) \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

The angle between AG and the downward vertical when the lamina first comes to rest is $\frac{\pi}{3}$.

c Equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-mg\frac{2}{3}h\sin\theta = \frac{5}{9}mh^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{6g}{5h}\sin\theta$$

The greatest magnitude of the angular acceleration corresponds to the greatest possible value of $\sin\theta$. In part **b**, you established that this is when $\theta = \frac{\pi}{3}$.

When

$$\theta = \frac{\pi}{3}$$

$$|\ddot{\theta}| = \frac{6g}{5h} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}g}{5h}$$

The greatest magnitude of the angular acceleration during the motion is $\frac{3\sqrt{3}g}{5h}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 31

Question:

To the end B of a thin uniform rod AB, of length $3a$ and mass m, is attached a thin uniform circular disc, of radius a and mass m, so that the rod and the diameter BC of the disc are in a straight line and $AC = 5a$.

a Show that the moment of inertia of this composite body, about an axis through A and perpendicular to AB and in the plane of the disc, is $\frac{77}{4}ma^2$.

The body is held at rest with the end A smoothly hinged to a fixed pivot and with the plane of the disc horizontal. The body is released and has angular speed ω when AC is vertical.

b Find ω in terms of a and g.

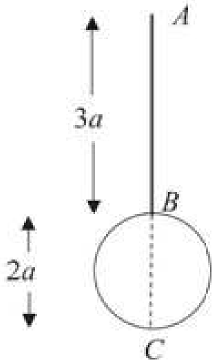
When AC is vertical, the centre of the disc strikes a stationary particle of mass $\frac{1}{2}m$.

Given that the particle adheres to the centre of the disc,

c show that the angular speed of the body immediately after impact is $\frac{77}{109}\omega$. ***E***

Solution:

a



The moment of inertia of the rod about the axis through A is given by

$$I_{\text{rod}} = \frac{4}{3}m\left(\frac{3a}{2}\right)^2 = 3ma^2$$

Using the standard result, $I = \frac{4}{3}ml^2$, for a rod of length $2l$ about an axis through its end, with $2l = 3a$.

By the parallel axes theorem, the moment of inertia of the disc about the axis through A is given by

$$I_{\text{disc}} = \frac{1}{4}ma^2 + m(4a)^2 = \frac{65}{4}ma^2$$

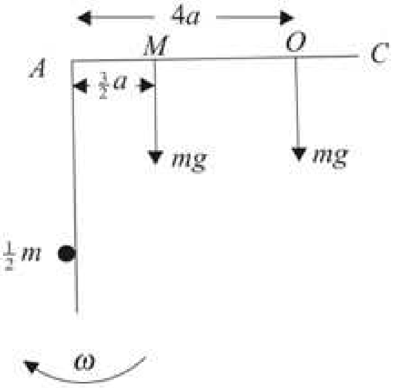
You can quote the result for the moment of inertia of a disc about its diameter and the centre of the disc is $4a$ from A.

The moment of inertia of the composite body is given by

$$I = I_{\text{rod}} + I_{\text{disc}}$$

$$= 3ma^2 + \frac{65}{4}ma^2 = \frac{77}{4}ma^2, \text{ as required}$$

b



If M is the centre of mass of the rod and O is the centre of the mass of the disc, then $AM = \frac{3}{2}a$ and $AO = 4a$.

Conservation of energy

$$\frac{1}{2}I\omega^2 = mg \times \frac{3}{2}a + mg \times 4a$$

$$\frac{77}{8}ma^2\omega^2 = \frac{11}{2}mga$$

$$\omega^2 = \frac{11mga}{2} \times \frac{8}{77ma^2} = \frac{4g}{7a}$$

$$\omega = \sqrt{\left(\frac{4g}{7a}\right)}$$

As the composite body moves from the horizontal to the vertical, the centre of mass, M, of the rod falls a vertical distance $\frac{3}{2}a$ and the centre of mass of the disc, O, falls a vertical distance $4a$.

c The moment of inertia, I', of the composite body and the particle of mass $\frac{1}{2}m$ about the axis through A is given by

$$I' = \frac{77}{4}ma^2 + \frac{1}{2}m(4a)^2 = \frac{109}{4}ma^2$$

Let ω' be the angular speed of the body immediately after impact.

Conservation of linear momentum about A

$$I'\omega' = I\omega$$

$$\frac{109}{4}ma^2\omega' = \frac{77}{4}ma^2\omega$$

$$\omega' = \frac{77}{109}\omega\text{, as required}$$

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Review Exercise 2
Exercise A, Question 32

Question:

a Prove, by integration, that the moment of inertia of a uniform rod, of mass m and length a, about an axis through its mid-point and perpendicular to the rod is $\frac{ma^2}{12}$.

Four uniform rods AB, BC, CD and DA, each of length a, are rigidly joined to form a square $ABCD$. Each of the rods AB, CD and DA has mass m and the rod BC has mass $3m$. The rods are free to rotate about a smooth horizontal axis L which passes through A and is perpendicular to the plane of the square.

b Show that the moment of inertia of the system about L is $6ma^2$ and find the distance of the centre of mass of the system from A.

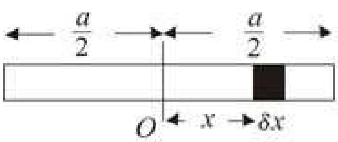
The system is released from rest with AB horizontal and C vertically below B.

c Find the greatest value of the angular speed of the system in the subsequent motion.

d Find the period of small oscillations of the system about the position of stable equilibrium. ***E***

Solution:

a



You consider the rod to be made up of a series of small pieces, or elements, each of length δx.

The mass per unit length of the rod is $\frac{m}{a}$.

When proving results you usually need to know the 'density' of the object, here the mass per unit length.

Consider an element of length δx at a distance x from the middle of the rod O.

$$\delta I = (\delta m)x^2 = \left(\frac{m}{a}\delta x\right)x^2 = \frac{mx^2}{a}\delta x$$

For the whole rod

$$I = \sum \delta I = \sum \frac{mx^2}{a}\delta x$$

The whole rod is the sum of the small pieces.

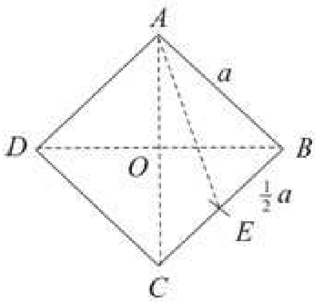
As $\delta x \to 0$

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{mx^2}{a}\,dx = \frac{m}{a}\left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{m}{a}\left[\frac{a^3}{24} - \left(-\frac{a^3}{24}\right)\right] = \frac{ma^2}{12}, \text{ as required}$$

As the small pieces range from one end of the rod to the other, x ranges from $-\frac{a}{2}$ at one end to $\frac{a}{2}$ at the other. So $-\frac{a}{2}$ and $\frac{a}{2}$ are the limits of the definite integral.

b



Let E be the mid-point of BC.

$$AE^2 = a^2 + \left(\frac{1}{2}a\right)^2 = \frac{5}{4}a^2$$

By the parallel axes theorem, the moment of inertia of the rod BC about the axis through A is given by

$$I_{BC} = \frac{1}{12}(3m)a^2 + (3m)AE^2$$

$$= \frac{1}{4}ma^2 + \frac{15}{4}ma^2 = 4ma^2$$

The mass of BC is $3m$.

Similarly for the rod CD

$I_{CD} = \frac{1}{12}ma^2 + m \times \frac{5}{4}a^2 = \frac{4}{3}ma^2$

As the mass of CD is one third of the mass of BC, this could just be written down.

The moments of inertia of the rods AB and AD about the axis through A are given by

$I_{AB} = I_{AD} = \frac{1}{12}ma^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{3}ma^2$

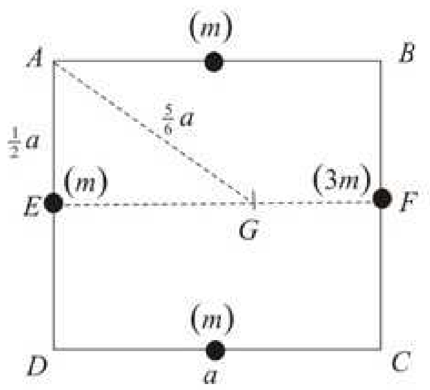
Using the parallel axes theorem.

The moment of inertia of the framework about the axis through A is given by

$I = I_{AB} + I_{BC} + I_{CD} + I_{DA}$

$= \frac{1}{3}ma^2 + 4ma^2 + \frac{4}{3}ma^2 + \frac{1}{3}ma^2$

$= 6ma^2$, as required



If E is the mid-point of AD and F is the mid-point of BC, then, by symmetry, the centre of mass of the framework G lies on EF. The location of G is found by taking moments about E.

M(E)

$6m \times EG = m \times \frac{a}{2} + 3m \times a + m \times \frac{a}{2} = 4ma$

The weight of each rod acts at the mid-point of the rod.

$EG = \frac{2}{3}a$

$AG^2 = AE^2 + EG^2 = \frac{1}{4}a^2 + \frac{4}{9}a^2 = \frac{25}{36}a^2$

$AG = \frac{5}{6}a$

The distance of the centre of mass of the system from A is $\frac{5}{6}a$.

c Let ω be the maximum angular speed of the system.
Conservation of energy

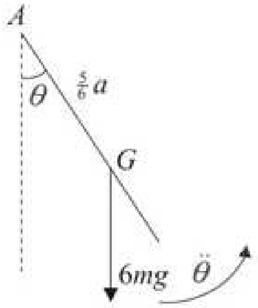
$$\frac{1}{2}I\omega^2 = 6mg\left(\frac{5}{6}a - \frac{1}{2}a\right)$$

$$3ma^2\omega^2 = 2mga$$

$$\omega^2 = \frac{2g}{3a} \Rightarrow \omega = \sqrt{\left(\frac{2g}{3a}\right)}$$

The maximum angular speed occurs when G is vertically below A. At that point G is $\frac{5}{6}a$ below A. Initially G is $\frac{1}{2}a$ below A. So G falls $\frac{5}{6}a - \frac{1}{2}a = \frac{1}{3}a$.

d



The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-6mg\left(\frac{5}{6}a\right)\sin\theta = 6ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{5g}{6a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$
Hence

$$\ddot{\theta} = -\frac{5g}{6a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{5g}{6a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{6a}{5g}\right)}$$

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Review Exercise 2
Exercise A, Question 33

Question:

A compound pendulum consists of a thin uniform rod AB, of length $2a$ and mass $3m$, with a particle of mass $2m$ attached at B. The pendulum is free to rotate in a vertical plane about a horizontal axis l which is perpendicular to the rod through a point C of the rod, where $AC = x, x < a$.

a Show that the moment of inertia of the pendulum about l is $(5x^2 - 14ax + 12a^2)m$.

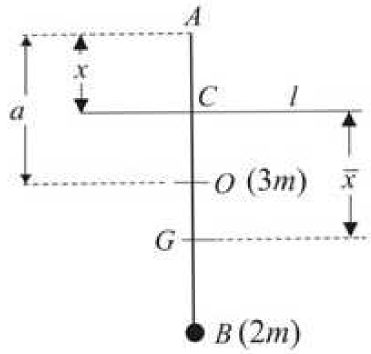
b Find the square of the period of small oscillations of the pendulum about l.

c Show that, as x varies, the period takes its minimum value when

$$x = \frac{(7-\sqrt{11})a}{5}.$$ ***E***

Solution:

a



Let O be the centre of the rod.
Using the parallel axes theorem, the moment of inertia of the rod about l is given by

$$I_{\text{rod}} = \frac{1}{3}(3m)a^2 + 3mOC^2$$

$$= ma^2 + 3m(a-x)^2$$

← $OC = a - x$

The moment of inertia of the compound pendulum about l is given by

$$I = I_{\text{rod}} + I_{\text{particle}}$$

$$= ma^2 + 3m(a-x)^2 + 2m(2a-x)^2$$

← $BC = 2a - x$

$$= ma^2 + 3ma^2 - 6max + 3mx^2 + 8ma^2 - 8max + 2mx^2$$

$$= 5mx^2 - 14max + 12ma^2$$

$$= (5x^2 - 14ax + 12a^2)m, \text{ as required}$$

b Let G be the centre of mass of the compound pendulum and $CG = \bar{x}$.

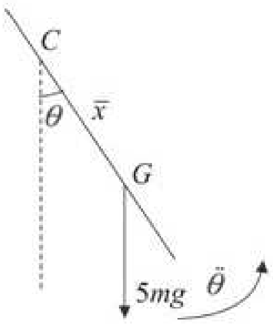
M(C)

$$5m\bar{x} = 3m(a-x) + 2m(2a-x)$$

$$= 7ma - 5mx$$

$$\bar{x} = \frac{7}{5}a - x$$

← You locate the position of G by taking the moments of mass about C.



The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-5mg\bar{x}\sin\theta = I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{5mg\bar{x}}{I}\sin\theta = -\frac{5mg\left(\frac{7}{5}a - x\right)}{(5x^2 - 14ax + 12a^2)m}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{(7a-5x)g}{5x^2-14ax+12a^2}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{(7a-5x)g}{5x^2-14ax+12a^2}$.

The period of small oscillations, T, is given by

$$T = \frac{2\pi}{\omega} \Rightarrow T^2 = \frac{4\pi^2}{\omega^2}$$

$$= \frac{4\pi^2}{g}\left(\frac{5x^2-14ax+12a^2}{7a-5x}\right)$$

c $$T^2 = \frac{4\pi^2}{g}\left(\frac{5x^2-14ax+12a^2}{7a-5x}\right)$$

$$2T\frac{dT}{dx} = \frac{4\pi^2}{g}\left[\frac{(7a-5x)(10x-14a)+5(5x^2-14ax+12a^2)}{(7a-5x)^2}\right]$$

Differentiate this equation throughout with respect to x, using implicit differentiation on the left hand side and the quotient rule on the right hand side.

At a minimum value $\frac{dT}{dx} = 0$

Hence

$$(7a-5x)(10x-14a)+5(5x^2-14ax+12a^2) = 0$$

$$70ax-98a^2-50x^2+70ax+25x^2-70ax+60a^2 = 0$$

$$-25x^2+70ax-38a^2 = 0$$

$$25x^2-70ax+38a^2 = 0$$

Using the quadratic formula $x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a}$.

$$x = \frac{70 \pm \sqrt{1100}}{50}a = \frac{7 \pm \sqrt{11}}{5}a$$

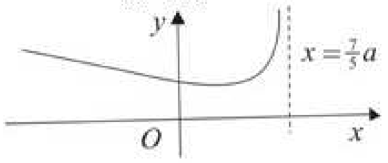
$x = \frac{7+\sqrt{11}}{5}a \approx 2.06a$ is impossible

This value is longer than the length of the rod, $2a$, and can be rejected.

The period takes its minimum value when

$x = \frac{(7-\sqrt{11})a}{5}$, as required

Unless the question specifically asks you do to so, you are not expected to show that the stationary point is a minimum. A sketch of $y = \frac{5x^2-14ax+12a^2}{7a-5x}$ is



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Review Exercise 2
Exercise A, Question 34

Question:

a Show, using integration, that the moment of inertia of a uniform equilateral triangular lamina, of side $2a$, and of mass m, about an axis through a vertex perpendicular to its plane is $\frac{5}{3}ma^2$.

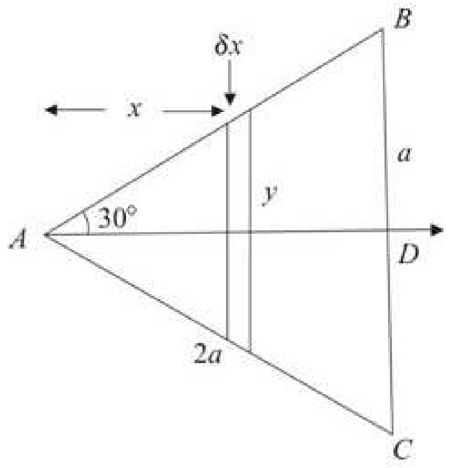
b Deduce that the moment of inertia of a uniform regular hexagonal lamina, of side $2a$ and mass M, about an axis through a vertex perpendicular to the plane of the lamina is $\frac{17}{3}Ma^2$.

A compound pendulum consists of a uniform regular hexagonal lamina $ABCDEF$, of side $2a$ and mass M, with a particle of mass $\frac{1}{2}M$ attached at the vertex D. The pendulum oscillates about a smooth horizontal axis which passes through the vertex A and is perpendicular to the plane of the lamina.

c Show that the period of small oscillations is $\pi\sqrt{\left(\frac{41a}{3g}\right)}$. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. The rod is a distance x from A.

Let the triangle be ABC and D the mid-point of BC as shown in the diagram above.

By Pythagoras' theorem

$$AD^2 = AB^2 - BD^2 = 4a^2 - a^2 = 3a^2$$

$$AD = \sqrt{3}a$$

The area of the triangle is given by

$$\frac{1}{2}BC \times AD = a \times \sqrt{3}a = \sqrt{3}a^2$$

To find an expression for the mass per unit area of the triangle, you have to first obtain an expression for the area of the triangle in terms of a.

The mass per unit area of the triangle is $\frac{m}{\sqrt{3}a^2}$

The moment of inertia, δI, of one elementary rod about the axis is given by

$$\delta I = \frac{1}{3}(\delta m)y^2 + (\delta m)x^2$$

$$= (\delta m)\left(\frac{1}{3}y^2 + x^2\right)$$

$$= \left(2y\delta x \times \frac{m}{\sqrt{3}a^2}\right)\left(\frac{1}{3}y^2 + x^2\right)$$

You use the standard formula for the moment of inertia of a rod about its centre and the parallel axes theorem to find the moment of inertia of a rod about the axis through the vertex A.

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By trigonometry

$$\frac{y}{x} = \tan 30° = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}x$$

Hence

$$\delta I = \left(2\frac{1}{\sqrt{3}}x\delta x \times \frac{m}{\sqrt{3}a^2}\right)\left(\frac{1}{3}\left(\frac{1}{\sqrt{3}}x\right)^2 + x^2\right)$$

$$= \frac{2mx}{3a^2}\delta x \times \left(\frac{1}{9}x^2 + x^2\right) = \frac{20m}{27a^2}x^3\delta x$$

$$I = \sum \delta I = \sum \frac{20m}{27a^2}x^3\delta x$$

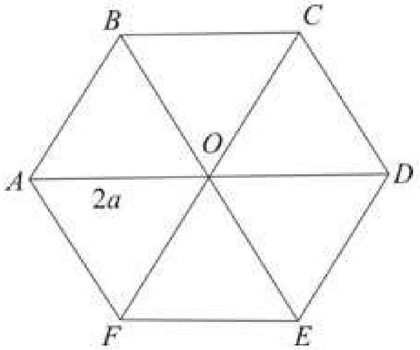
As $\delta x \to 0$

$$I = \frac{20m}{27a^2}\int_0^{\sqrt{3}a} x^3 \, dx = \frac{20m}{27a^2}\left[\frac{x^4}{4}\right]_0^{\sqrt{3}a}$$

$$= \frac{20m}{27a^2} \times \frac{9a^4}{4} = \frac{5}{3}ma^2, \text{ as required}$$

The upper limit corresponds to the distance AD which is $\sqrt{3}a$.

b



You consider the hexagon to be made up of 6 triangles. Each of the six triangles has the moment of inertia found in part **a** about the centre of the hexagon O.

The hexagon is made up of 6 triangles of mass m, where $M = 6m$.
The moment of inertia of the hexagon about an axis through O, I_O, is given by

$$I_O = 6 \times \frac{5}{3}ma^2 = \frac{5}{3}Ma^2$$

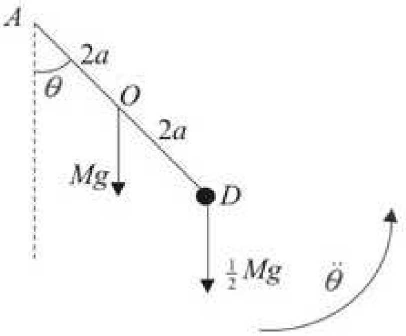
As $M = 6m$.

By the parallel axes theorem, the moment of inertia of the hexagon about an axis through A, I_A, is given by

$$I_A = I_O + M \times OA^2$$

$$= \frac{5}{3}Ma^2 + M \times 4a^2 = \frac{17}{3}Ma^2, \text{ as required}$$

c



The moment of inertia of the compound pendulum, I, about the axis through A is given by

$I = I_{\text{hexagon}} + I_{\text{particle}}$

$= \frac{17}{3}Ma^2 + \frac{1}{2}M(4a)^2 = \frac{41}{3}Ma^2$

The equation of angular motion about A is

$$L = I\ddot{\theta}$$

$$-Mg \times 2a\sin\theta - \frac{1}{2}Mg \times 4a\sin\theta = \frac{41}{3}Ma^2\ddot{\theta}$$

$$-4Mga\sin\theta = \frac{41}{3}Ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{12g}{41a}\sin\theta$$

For small $\theta, \sin\theta \approx \theta$

Hence

$\ddot{\theta} = -\frac{12g}{41a}\theta$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{12g}{41a}$.

The period of small oscillations is given by

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{41a}{12g}\right)} = \pi\sqrt{\left(\frac{41a}{3g}\right)}$, as required

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Review Exercise 2
Exercise A, Question 35

Question:

a Show, by integration, that the moment of inertia of a uniform circular disc, of mass m and radius a, about an axis through the centre and perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

b Deduce the moment of inertia of the disc about a diameter.

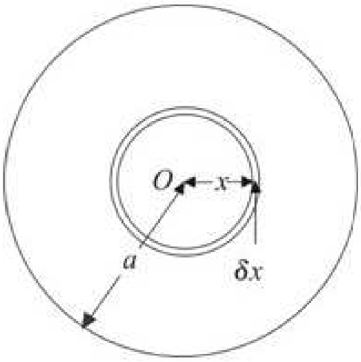
c Show that the moment of inertia of a uniform right circular cone, of height r, base radius r and mass M about an axis through its vertex and parallel to a diameter of the base is $\frac{3}{4}Mr^2$.

The above cone is free to turn about a fixed smooth pivot at its vertex and is released from rest with its axis horizontal.

d Find the angular speed of the cone when its axis is vertical. ***E***

Solution:

a



You consider the disc as made up a series of concentric rings of thickness δx. The area of each ring is $2\pi x\delta x$.

The mass per unit area of the disc is $\frac{m}{\pi a^2}$.

The moment of inertia, δI, of a ring is given by

$$\delta I = (\delta m)x^2 = \left(2\pi x\delta x \times \frac{m}{\pi a^2}\right)x^2 = \frac{2m}{a^2}x^3\delta x$$

The moment of inertia of the disc, I, is given by

$$I = \sum \delta I = \sum \frac{2m}{a^2}x^3\delta x$$

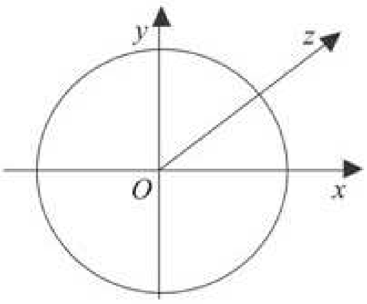
As $\delta x \to 0$

$$I = \int_0^a \frac{2m}{a^2}x^3\,dx = \frac{2m}{a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{2m}{a^2}\left(\frac{a^4}{4} - 0\right)$$

$$= \frac{1}{2}ma^2\text{, as required}$$

This is a standard result which you should be able to prove. You are expected to be able to prove all of the standard results given in the Formulae Booklet and you should practise writing these out.

b



Let the centre of the disc be O and Ox and Oy be perpendicular axes through O.
By the perpendicular axes theorem, the moment of inertia, I_{Ox}, about a diameter Ox through O is given by

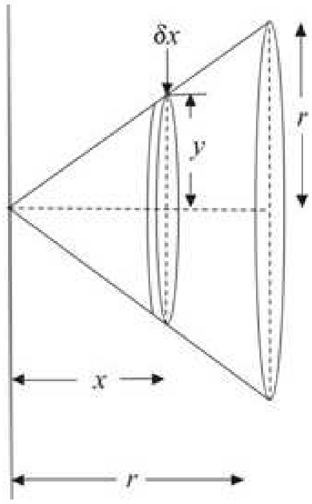
$$I_{Ox} + I_{Oy} = I_{Oz}$$

$$2I_{Ox} = \frac{1}{2}ma^2$$

$$I_{Ox} = \frac{1}{4}ma^2$$

By symmetry the moment of inertia about the axis Ox equals the moment of inertia about the axis Oy.

c Axis of rotation



You consider the cone to be made up of thin discs, each of thickness δx with the centre of the disc at a distance x from the vertex of the cone. If the radius of the disc is y, then, using the formula, $V = \pi r^2 h$, for the volume of a cylinder, the volume of a thin disc is $\pi y^2 \delta x$.

The mass per unit volume of the cone is

$$\frac{M}{\frac{1}{3}\pi r^2 \times r} = \frac{3M}{\pi r^3}$$

Using the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, with $h = r$.

The moment of inertia of an elementary disc about the axis of rotation is given by

$$\delta I = \frac{1}{4}(\delta m)y^2 + (\delta m)x^2$$

You use the answer to part **b** with $m = \delta m$ and $a = y$, and the parallel axes theorem to form an expression for the moment of inertia, δI, of the thin disc.

By similar triangles

$$\frac{y}{x} = \frac{r}{r} \Rightarrow y = x$$

Hence

$$\delta I = \frac{1}{4}(\pi y^2 \delta x)\left(\frac{3M}{\pi r^3}\right)y^2 + (\pi y^2 \delta x)\left(\frac{3M}{\pi r^3}\right)x^2$$

$$= \frac{3M}{r^3}\left(\frac{y^4}{4} + y^2 x^2\right)\delta x = \frac{3M}{r^3}\left(\frac{x^4}{4} + x^4\right)\delta x$$

$$= \frac{15M}{4r^3}x^4 \delta x$$

The mass, δm, of the disc is its volume, $\pi y^2 \delta x$, multiplied by the mass per unit volume $\frac{3M}{\pi r^3}$.

As $y = x$.

For the complete cone

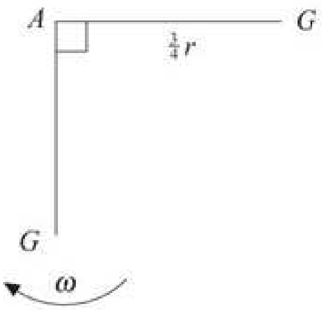
$$I = \sum \delta I = \sum \frac{15M}{4r^3}x^4 \delta x$$

As $\delta x \to 0$

$$I = \frac{15M}{4r^3}\int_0^r x^4 \, dx = \frac{15M}{4r^3}\left[\frac{x^5}{5}\right]_0^r$$

$$= \frac{15M}{4r^3}\left(\frac{r^5}{5} - 0\right) = \frac{3}{4}Mr^2, \text{ as required}$$

d



Let the vertex of the cone be A and the centre of mass of the cone be G, then

$AG = \frac{3}{4}r$.

Let the angular speed of the cone when its axis is vertical be ω.

Conservation of energy

$$\frac{1}{2}I\omega^2 = Mg \times \frac{3}{4}r$$

$$\frac{3}{8}Mr^2\omega^2 = \frac{3}{4}Mgr \Rightarrow \omega^2 = \frac{2g}{r}$$

$$\omega = \sqrt{\left(\frac{2g}{r}\right)}$$

> The standard result for the centre of mass of a cone can be found among the formulae for module M3 in the Formulae Booklet.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 36

Question:

a Find the moment of inertia of a uniform square lamina $ABCD$, of side $2a$ and mass m, about an axis through A perpendicular to the plane of the lamina.

The lamina is free to rotate about a fixed smooth horizontal axis through A perpendicular to the plane of the lamina.

b Show that the period of small oscillations about the stable equilibrium position is

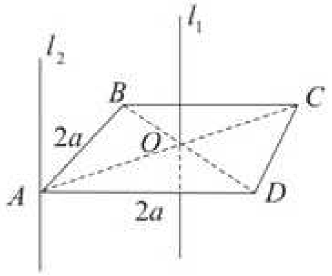
$$2\pi\left(\frac{8a}{3g\sqrt{2}}\right)^{\frac{1}{2}}.$$

The lamina is rotating with angular speed ω when C is vertically below A.

c Determine the components, along and perpendicular to AC, of the reaction of the lamina on the axis when AC makes an angle θ with the downward vertical through A. ***E***

Solution:

a



Let O be the centre of the lamina
By Pythagoras

$AO^2 + BO^2 = (2a)^2$

$2AO^2 = 4a^2 \Rightarrow AO^2 = 2a^2 \Rightarrow AO = \sqrt{2}a$

As $AO = BO$.

Let l_1 be the axis through O perpendicular to the plane of the lamina and l_2 be the axis through A perpendicular to the plane of the lamina.
The moment of inertia of the lamina about l_1 is given by

$$I_{l_1} = \frac{1}{3}m(a^2 + a^2) = \frac{2}{3}ma^2.$$

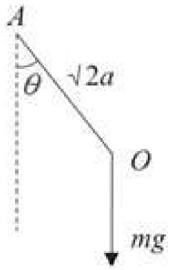
The Formulae Booklet gives you that the moment of inertia of a rectangle, mass m, sides $2a$ and $2b$, about a perpendicular axis through its centre is $\frac{1}{3}m(a^2 + b^2)$ and, for a square, $a = b$.

By the parallel axis theorem, the moment of inertia of the lamina about l_2 is given by

$$I_{l_2} = I_{l_1} + mAO^2$$

$$= \frac{2}{3}ma^2 + m \times 2a^2 = \frac{8}{3}ma^2$$

b



Equation of angular motion about l_2

$$L = I\ddot{\theta}$$

$$-mg \times \sqrt{2}a \sin\theta = \frac{8}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{3g\sqrt{2}}{8a}\sin\theta \quad ①$$

You will need this equation to find the component of the reaction perpendicular to AC in part **c**.

For small θ, $\sin\theta \approx \theta$

Hence

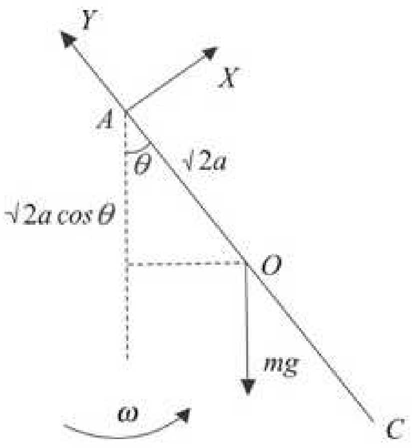
$$\ddot{\theta} = -\frac{3g\sqrt{2}}{8a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{3g\sqrt{2}}{8a}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{8a}{3g\sqrt{2}}\right)^{\frac{1}{2}}, \text{ as required}$$

c



Let X and Y be the components, perpendicular to and along AC, of the reaction of the lamina on the axis.

Conservation of energy

$$\frac{1}{2}I\omega^2 - \frac{1}{2}I\dot{\theta}^2 = mg(\sqrt{2}a - \sqrt{2}a\cos\theta)$$

$$\frac{4}{3}ma^2\left(\omega^2 - \dot{\theta}^2\right) = mg\sqrt{2}a(1-\cos\theta)$$

$$\omega^2 - \dot{\theta}^2 = \frac{3g\sqrt{2}}{4a}(1-\cos\theta)$$

$$\dot{\theta}^2 = \omega^2 - \frac{3g\sqrt{2}}{4a}(1-\cos\theta) \qquad ②$$

R($\parallel AC$)

$$Y - mg\cos\theta = m\left(\sqrt{2}a\right)\dot{\theta}^2$$

$$Y = mg\cos\theta + m\sqrt{2}a\left(\omega^2 - \frac{3g\sqrt{2}}{4a}(1-\cos\theta)\right)$$

Using equation ②.

$$= mg\cos\theta + m\sqrt{2}a\omega^2 - \frac{6mg}{4}(1-\cos\theta)$$

$$= ma\sqrt{2}\omega^2 + \frac{5mg}{2}\cos\theta - \frac{3mg}{2}$$

No further simplification of this expression is possible.

R($\perp AC$)

$$X - mg\sin\theta = m(\sqrt{2}a)\ddot{\theta}$$

$$= -m(\sqrt{2}a)\frac{3g\sqrt{2}}{8a}\sin\theta = -\frac{3}{4}mg\sin\theta$$

Using equation ① in part **b**.

$$X = \frac{1}{4}mg\sin\theta$$

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 37

Question:

A uniform lamina of mass M is in the form of an isosceles triangle ABC, with $AB = AC = 5l$ and $BC = 6l$.

a Show, by integration, that the moment of inertia of the lamina about an axis which passes through A and is parallel to BC is $8Ml^2$.

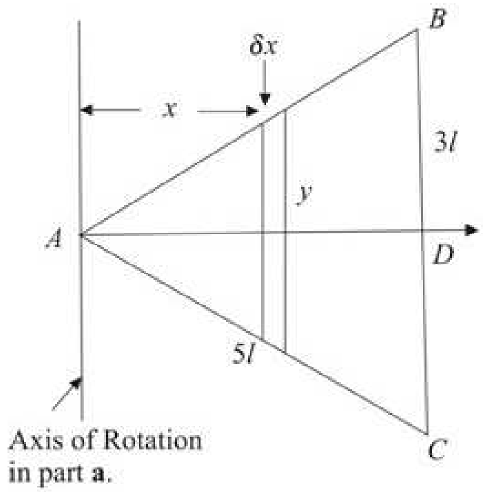
b Find also the moment of inertia of the lamina about an axis that passes through A and the mid-point of BC.

A particle of mass M is attached to the lamina at the mid-point of BC. The system is free to rotate about a smooth horizontal axis through A perpendicular to the plane of the triangle.

c Find the period of small oscillations about the position of equilibrium in which BC is below A. ***E***

Solution:

a



You consider the triangle as made up of thin rods of length $2y$, thickness δx and mass δm. Each rod is a distance x from A.

Let the mid-point of BC be D.
By Pythagoras

$$AD^2 = AB^2 - BD^2 = (5l)^2 - (3l)^2 = 16l^2 \Rightarrow AD = 4l$$

The area of $\Delta ABC = \frac{1}{2} BC \times AD = 3l \times 4l = 12l^2$

The mass per unit area of the triangle is $\frac{M}{12l^2}$

The moment of inertia, δI, of one elementary rod about the axis is given by

$$\delta I = (\delta m)x^2$$

$$= \left(2y\delta x \times \frac{M}{12l^2}\right)x^2 = \frac{M}{6l^2}x^2 y \delta x$$

All points on the rod are a distance x from the axis of rotation.

The mass, δm, of an elementary rod is its area, $2y\delta x$, multiplied by the mass per unit area.

By similar triangles

$$\frac{y}{x} = \frac{3l}{4l} \Rightarrow y = \frac{3}{4}x$$

Hence

$$\delta I = \frac{M}{6l^2}x^2 \times \frac{3}{4}x\delta x = \frac{M}{8l^2}x^3 \delta x$$

$$I = \sum \delta I = \sum \frac{M}{8l^2}x^3 \delta x$$

As $\delta x \to 0$

$$I = \frac{M}{8l^2}\int_0^{4l} x^3 \, dx = \frac{M}{8l^2}\left[\frac{x^4}{4}\right]_0^{4l}$$

$$= \frac{M}{8l^2} \times \frac{256l^4}{4} = 8Ml^2, \text{ as required}$$

The upper limit corresponds to the distance AD which is $4l$.

b The moment of inertia, δI, of one elementary rod about AD is given by

$$\delta I = \frac{1}{3}(\delta m)y^2$$

Using the standard result for the moment of inertia of a rod about an axis through its centre.

$$= \frac{1}{3}\left(2y\delta x \times \frac{M}{12l^2}\right)y^2 = \frac{M}{18l^2}y^3\delta x$$

$$= \frac{M}{18l^2} \times \frac{27}{64}x^3\delta x = \frac{3M}{128l^2}x^3\delta x$$

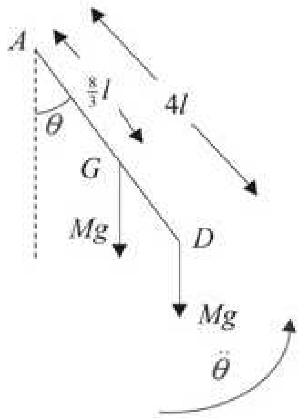
$$I = \sum \delta I = \sum \frac{3M}{128l^2}x^3\delta x$$

As $\delta x \to 0$

$$I = \frac{3M}{128l^2}\int_0^{4l} x^3 \, \mathrm{d}x = \frac{3M}{128l^2}\left[\frac{x^4}{4}\right]_0^{4l}$$

$$= \frac{3M}{128l^2} \times \frac{256l^4}{4} = \frac{3}{2}Ml^2$$

c



Using the standard result that the centre of mass of a triangle is $\frac{2}{3}$ along the median from the vertex, if G is the centre of mass of the triangle then $AG = \frac{2}{3} \times 4l = \frac{8}{3}l$.

The moment of inertia, I, of the triangle about a smooth horizontal axis through A perpendicular to the plane of the triangle is given by

$$I = 8Ml^2 + \frac{3}{2}Ml^2 = \frac{19Ml^2}{2}$$

The axis of rotation in part **c** is perpendicular to both the axes in part **a** and part **b**. So you find the moment of inertia required for part **c** using the perpendicular axes theorem.

The moment of inertia, I', of the triangle and particle about the axis is given by

$$I' = \frac{19}{2}Ml^2 + M(4l)^2 = \frac{51}{2}Ml^2$$

Equation of angular motion about the axis through A perpendicular to the plane of the triangle

$$L = I\ddot{\theta}$$

$$-Mg \times \frac{8}{3}l\sin\theta - Mg \times 4l\sin\theta = \frac{51}{2}Ml^2\ddot{\theta}$$

$$-\frac{20}{3}Mgl\sin\theta = \frac{51}{2}Ml^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{40g}{153l}\sin\theta$$

You can take the moments of the weights for the triangle and particle about the axis separately.

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{40g}{153l}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{40g}{153l}$.

The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{153l}{40g}\right)} = \pi\sqrt{\left(\frac{153l}{10g}\right)}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 38

Question:

A body consists of 2 uniform discs, each of mass m and radius a, the centres of which are fixed to the ends A and B of a uniform rod of mass m and length $5a$. The discs and the rod are coplanar. The body is free to rotate about a fixed smooth horizontal axis which is perpendicular to the plane of the dics and which passes through O on the rod where $OA = 2a$.

a Show that the moment of inertia of the body about this axis is $\frac{49}{3}ma^2$.

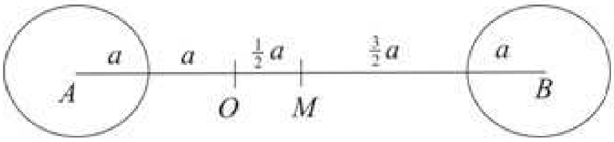
The body is initially at rest with B vertically below O. A particle of mass $6m$ is moving horizontally in the plane of the disc with speed u. It strikes the rod at a point P below O, where $OP = x$, and adheres to the rod.

b Find, in terms of a, x and u, the angular speed with which the system starts to move immediately after impact.

c Show that this angular speed is a maximum when $x = \frac{7}{6}a\sqrt{2}$.

d Given that $x = \frac{4}{3}a$, find the value of u for which the rod just reaches the horizontal position in the subsequent motion. ***E***

Solution:



Let the mid-point of AB be M.

By the parallel axes theorem, the moment of inertia, I_{rod}, about O is given by

$$I_{\text{rod}} = \frac{1}{3}m\left(\frac{5a}{2}\right)^2 + mOM^2 = \frac{25}{12}ma^2 + m\left(\frac{a}{2}\right)^2$$

$$= \frac{7}{3}ma^2$$

Using the standard result, $I = \frac{1}{3}ml^2$, for a rod of length $2l$ about an axis through its centre, with $2l = 5a$.

By the parallel axes theorem, the moment of inertia, I_A, of the disc centre A about O is given by

$$I_A = \frac{1}{2}ma^2 + mAO^2 = \frac{1}{2}ma^2 + m(2a)^2$$

$$= \frac{9}{2}ma^2$$

By the parallel axes theorem, the moment of inertia, I_B, of the disc centre B about O is given by

$$I_B = \frac{1}{2}ma^2 + mBO^2 = \frac{1}{2}ma^2 + m(3a)^2$$

$$= \frac{19}{2}ma^2$$

$BO = 5a - OA = 5a - 2a = 3a$

The moment of inertia, I, of the body about O is given by

$$I = I_{\text{rod}} + I_A + I_B$$

$$= \frac{7}{3}ma^2 + \frac{9}{2}ma^2 + \frac{19}{2}ma^2 = \frac{49}{3}ma^2, \text{ as required}$$

b The moment inertia, I', of the body together with the particle about O is given by

$$I' = I + 6mx^2 = \frac{49}{3}ma^2 + 6mx^2$$

Let the angular speed with which the system starts to move be ω.

Conservation of linear momentum about O

$$6mux = I'\omega = \left(\frac{49}{3}ma^2 + 6mx^2\right)\omega$$

$$\omega = \frac{6mux}{\frac{49}{3}ma^2 + 6mx^2} = \frac{18ux}{49a^2 + 18x^2}$$

c $\frac{d\omega}{dx} = 18u\left[\frac{49a^2 + 18x^2 - x \times 36x}{\left(49a^2 + 18x^2\right)^2}\right] = \frac{18u}{\left(49a^2 + 18x^2\right)^2}(49a^2 - 18x^2)$

$\frac{d\omega}{dx} = 0 \Rightarrow 49a^2 - 18x^2 = 0$

$x^2 = \frac{49}{18}a^2 = \frac{49 \times 2}{36}a^2$

$x = \frac{7}{6}a\sqrt{2}$, as required

Unless you are asked specifically to do so, you are not expected to show that the stationary point is a maximum. If you were asked to do so, you could argue that as $18x^2$ ranges from less than $49a^2$ to more than $49a^2$, $\frac{d\omega}{dx}$ changes sign from positive to negative and, so, the point is a maximum.

d If $x = \frac{4}{3}a$, $\omega = \frac{18u \times \frac{4}{3}a}{49a^2 + 18\left(\frac{4}{3}a\right)^2} = \frac{24ua}{81a^2} = \frac{8u}{27a}$

and $I' = \frac{49}{3}ma^2 + 6m\left(\frac{4}{3}a\right)^2$

$= \frac{49}{3}ma^2 + \frac{32}{3}ma^2 = 27ma^2$

Conservation of energy

$\frac{1}{2}I\omega^2 = 3mg \times \frac{1}{2}a + 6mgx$

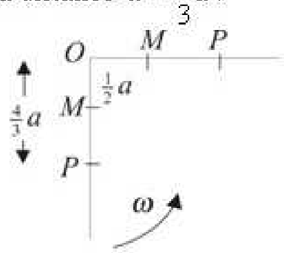
$\frac{1}{2} \times 27ma^2\left(\frac{8u}{27a}\right)^2 = 3mg\left(\frac{1}{2}a\right) + 6mg\left(\frac{4}{3}a\right) = \frac{19}{2}mga$

$\frac{32}{27}mu^2 = \frac{19}{2}mga$

$u^2 = \frac{513}{64}ga = \frac{9 \times 57}{64}ga$

$u = \frac{3}{8}\sqrt{(57ga)}$

To reach the horizontal, the centre of mass M of the body must rise a distance $\frac{1}{2}a$ and the particle must rise a distance $x = \frac{4}{3}a$.



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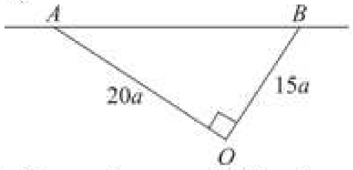
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 39

Question:

a Use integration to show that the moment of inertia of a uniform rod of mass m and length l, about an axis through one end and inclined at an angle θ to the rod is $\frac{1}{3}ml^2\sin^2\theta$.



The figure shows a rigid body consisting of two uniform rods. Rod AO has mass m and length $20a$ and the rod BO has mass m and length $15a$. They are rigidly joined together at O so that angle AOB is a right angle. The body is free to rotate about a fixed horizontal axis AB and hangs in equilibrium with O below AB. A particle of mass $\frac{1}{3}m$ is moving horizontally at right angles to the plane OAB with speed u. It collides with, and adheres to, the body at O.

b Show that the moment of inertia of the composite body, consisting of the two rods and the particle, about AB is $144ma^2$.

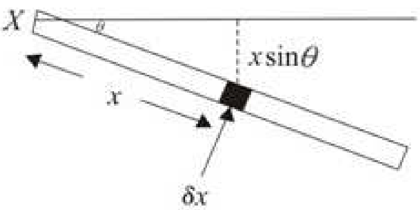
c Find the range of values of u for which the composite body will make complete revolutions.

Given that the composite body does make complete revolutions,

d find the value of u for which the greatest angular speed during the subsequent motion is twice the smallest angular speed. ***E***

Solution:

a



The mass per unit area is $\frac{m}{l}$.

Consider an element of length δx at a distance x from one end of the rod X.

$$\delta I = (\delta m)(x\sin\theta)^2 = \left(\frac{m}{l}\delta x\right)x^2\sin^2\theta$$

Each elementary particle, of mass δm, is at a distance $x\sin\theta$ from the axis of rotation.

$$= \frac{m\sin^2\theta}{l}x^2\delta x$$

For the whole rod

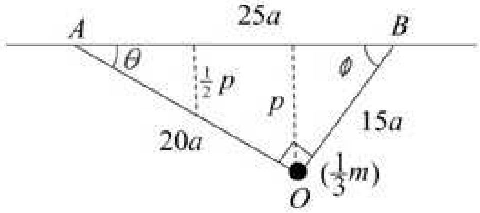
$$I = \sum \delta I = \sum \frac{m\sin^2\theta}{l}x^2\delta x$$

As $\delta x \to 0$

$$I = \int_0^l \frac{m\sin^2\theta}{l}x^2\,dx = \frac{m\sin^2\theta}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{m\sin^2\theta}{l}\left[\frac{l^3}{3} - 0\right] = \frac{1}{3}ml^2\sin^2\theta, \text{ as required}$$

b



$$AB^2 = (20a)^2 + (15a)^2 = (25a)^2 \Rightarrow AB = 25a$$

$$\sin\theta = \frac{15a}{25a} = \frac{3}{5}, \sin\phi = \frac{20a}{25a} = \frac{4}{5}$$

$$\frac{p}{20a} = \sin\theta = \frac{3}{5} \Rightarrow p = 12a$$

Part **a** gives you the clue that you need to start by finding the sines of the angles the rods make with AB.

The moment of inertia of the composite body is given by

$$I = I_{\text{rod } AO} + I_{\text{rod } OB} + I_{\text{particle}}$$

$$= \frac{1}{3}m(20a)^2\sin^2\theta + \frac{1}{3}m(15a)^2\sin^2\phi + \left(\frac{1}{3}m\right)p^2$$

$$= \frac{1}{3}m \times 400a^2 \times \frac{9}{25} + \frac{1}{3}m \times 225a^2 \times \frac{16}{25} + \frac{1}{3}m \times 144a^2$$

$$= 48ma^2 + 48ma^2 + 48ma^2 = 144ma^2, \text{ as required}$$

c Let ω be the angular speed of the composite body immediately after impact

Conservation of angular momentum about AB

$$\frac{1}{3}mu \times 12a = I\omega$$

$$4mua = 144ma^2\omega$$

$$\omega = \frac{4mua}{144ma^2} = \frac{u}{36a}$$

Using energy, for complete revolutions

$$\frac{1}{2}I\omega^2 > 2mgp + \frac{2}{3}mgp$$

$$72ma^2\omega^2 > \frac{8}{3}mgp$$

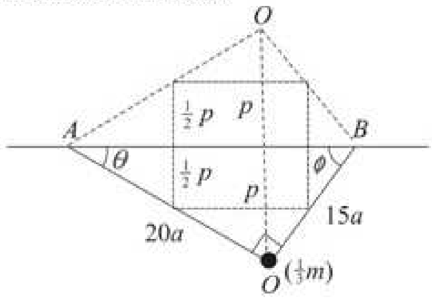
$$72ma^2\left(\frac{u}{36a}\right)^2 > \frac{8}{3}mg \times 12a$$

$$\frac{ma^2u^2}{18a^2} > 32mga$$

$$u^2 > 576ga$$

$$u > 24\sqrt{(ga)}$$

For complete revolutions, there must be enough initial kinetic energy to raise each of the centres of mass of the rods a vertical distance of $2 \times \frac{1}{2}p$ and the particle a distance of $2 \times p$.



d The greatest angular speed is immediately after the impact and, from part **c**, is $\frac{u}{36a}$.

The least angular speed is when O is vertically above AB and is $\frac{1}{2} \times \frac{u}{36a} = \frac{u}{72a}$

Conservation of energy

$$\frac{1}{2}I\left(\frac{u}{36a}\right)^2 - \frac{1}{2}I\left(\frac{u}{72a}\right)^2 = 32mga$$

The increase in potential energy needed for complete revolutions is the same as in part **c**.

$$72ma^2\left(\frac{u^2}{36^2a^2} - \frac{u^2}{72^2a^2}\right) = 32mga$$

$$\frac{1}{24}mu^2 = 32mga$$

$$u^2 = 24 \times 32ga = 256 \times 3ga$$

$$u = 16\sqrt{(3ga)}$$

Solutionbank M5

Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 40

Question:

A uniform right angled triangular lamina ABM of mass m is such that $\angle AMB = 90°$, $AM = 3a$ and $BM = a$.

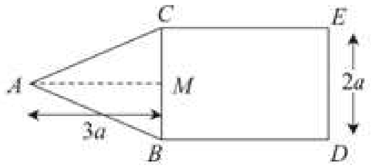
a Find the moment of inertia of the lamina about

i BM,

ii an axis through A parallel to BM,

iii AM.

b Deduce that the moment of inertia of a uniform triangular lamina ABC of mass $2m$ such that $AB = AC, BC = 2a$ and $AM = 3a$, where M is the mid-point of BC, about an axis through A perpendicular to the plane of the lamina is $\frac{28}{3}ma^2$.



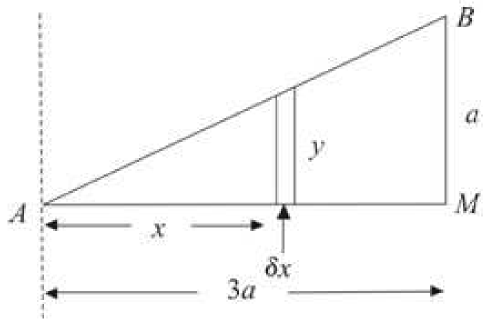
The figure shows a uniform plane lamina of mass $6m$ formed by joining a triangular lamina ABC to a rectangular lamina $BDEC$ along a common line BC. The sides BD and CE are each of length $3a$. The side DE is of length $2a$. $AB = AC$ and $AM = 3a$, where M is the mid-point of BC. The lamina can rotate about a smooth fixed axis through A, perpendicular to the plane of the lamina.

c Show that the moment of inertia of the lamina about this axis is $\frac{284}{3}ma^2$.

d Determine the period of small oscillations of the lamina about its position of stable equilibrium. ***E***

Solution:

a



Area of triangle $AMB = \frac{1}{2}3a \times a = \frac{3}{2}a^2$

Mass per unit area is $\frac{m}{\frac{3}{2}a^2} = \frac{2m}{3a^2}$

By similar triangles

$\frac{y}{x} = \frac{a}{3a} \Rightarrow y = \frac{1}{3}x$

i About BM

$\delta I = (\delta m)(3a-x)^2 = \left(y\delta x \times \frac{2m}{3a^2}\right)(3a-x)^2$ ← Each point on the thin strip is a distance $(3a-x)$ from BM.

$= \frac{2m}{9a^2}x(3a-x)^2\delta x$ ← Using $y = \frac{1}{3}x$.

Hence

$$I = \frac{2m}{9a^2}\int_0^{3a} x(3a-x)^2\,dx = \frac{2m}{9a^2}\int_0^{3a}(9a^2x - 6ax^2 + x^3)\,dx$$

$$= \frac{2m}{9a^2}\left[\frac{9a^2x^2}{2} - 2ax^3 + \frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{9a^2}\left[\frac{81a^4}{2} - 54a^4 + \frac{81a^4}{4}\right] = \frac{2m}{9a^2} \times \frac{27a^4}{4}$$

$$= \frac{3}{2}ma^2$$

ii About an axis through A parallel to BM

$\delta I = (\delta m)x^2 = \left(y\delta x \times \frac{2m}{3a^2}\right)x^2$ ← This axis is shown by a dotted line on the diagram above.

$= \frac{2m}{9a^2}x^3\delta x$

Hence

$$I = \frac{2m}{9a^2}\int_0^{3a} x^3 \, dx = \frac{2m}{9a^2}\left[\frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{9a^2} \times \frac{81a^4}{4} = \frac{9}{2}ma^2$$

iii About AM

$$\delta I = \frac{4}{3}(\delta m)\left(\frac{y}{2}\right)^2 = \frac{4}{3}\left(y\delta x \times \frac{2m}{3a^2}\right)\left(\frac{y}{2}\right)^2$$

$$= \frac{2m}{9a^2}y^3\delta x = \frac{2m}{243a^2}x^3\delta x$$

Using the standard result, $I = \frac{4}{3}ml^2$, for a rod of length $2l$ about an axis through its end, with $m = \delta m$ and $2l = y$.

Using $y = \frac{1}{3}x$.

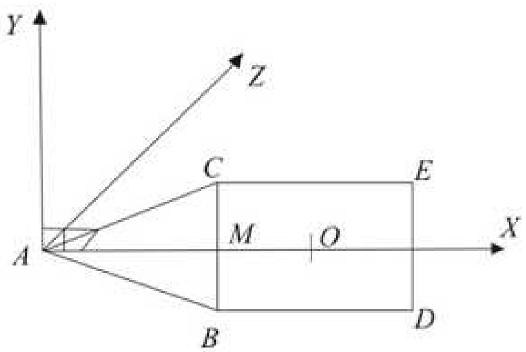
Hence

$$I = \frac{2m}{243a^2}\int_0^{3a} x^3 \, dx = \frac{2m}{243a^2}\left[\frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{2m}{243a^2} \times \frac{81a^4}{4} = \frac{1}{6}ma^2$$

You can deduce this result, without integrating, from the answer to part **a i**. Replacing $3a$ by a, $I = \frac{3}{2}m\left(\frac{a}{3}\right)^2 = \frac{1}{6}ma^2$.

b For this part of the question, just consider triangle ABC in the diagram below. The full diagram is needed for part **c**.



The axis of rotation, AZ, is perpendicular to the plane of the lamina, that is, it is perpendicular to both AX and AY.

The moment of inertia of the two triangles about AX is given by

$$I_{AX} = 2 \times \frac{1}{6}ma^2 = \frac{1}{3}ma^2$$

Note that the triangle ABC has mass $2m$, so each right angled triangle has mass m.

That is twice the answer to part **a iii**.

The moment of inertia of the two triangles about AY is given by

$$I_{AY} = 2 \times \frac{9}{2}ma^2 = 9ma^2$$

That is twice the answer to part **a ii**.

By the perpendicular axes theorem, the moment of inertia of the two triangles about AZ is given by

$$I_{\text{triangles}} = I_{AX} + I_{AY} = \frac{1}{3}ma^2 + 9ma^2 = \frac{28}{3}ma^2$$

c The moment of inertia of the rectangle about an axis through its centre O perpendicular to the plane of the lamina is given by

The area of each triangle ACM and ABM is one quarter of the area of the rectangle $BDEC$. As the total mass is $6m$, the mass of each triangle is m and the mass of the rectangle is $4m$.

$$I_O = \frac{1}{3}(4m)\left(a^2 + \left(\frac{3a}{2}\right)^2\right) = \frac{13}{3}ma^2$$

Using the standard result that the moment of inertia of a rectangle, sides $2a$ and $2b$, about a perpendicular axis through its centre is $\frac{1}{3}m\left(a^2+b^2\right)$ with $2b = 3a$. The mass of this rectangle is $4m$.

By the parallel axes theorem, the moment of inertia of the rectangle about AZ is given by

$$I_{\text{rectangle}} = I_O + 4mOA^2$$

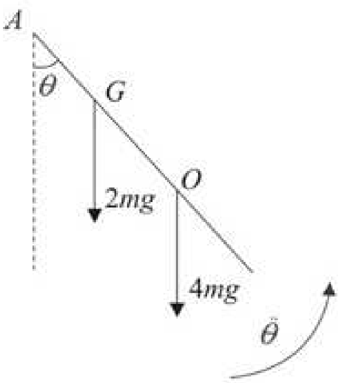
$$= \frac{13}{3}ma^2 + 4m\left(\frac{9a}{2}\right)^2 = \frac{256}{3}ma^2$$

The moment of inertia of the complete lamina about AZ is given by

$$I = I_{\text{triangles}} + I_{\text{rectangle}}$$

$$= \frac{28}{3}ma^2 + \frac{256}{3}ma^2 = \frac{284}{3}ma^2$$, as required

d



If G is the centre of mass of the triangle ABC, then, as AM is the median of the triangle,
$AG = \frac{2}{3}AM = \frac{2}{3} \times 3a = 2a$.

Equation of angular motion about AZ.

$$L = I\ddot{\theta}$$

$$-2mg \times AG \sin\theta - 4mg \times AO \sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$-2mg \times 2a \sin\theta - 4mg \times \frac{9a}{2}\sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$-22mga \sin\theta = \frac{284}{3}ma^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{33g}{142a}\sin\theta$$

For small θ, $\sin\theta \approx \theta$

Hence

$$\ddot{\theta} = -\frac{33g}{142a}\theta$$

Comparing with the standard equation for simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, the motion is approximately simply harmonic, with $\omega^2 = \frac{33g}{142a}$. The period of small oscillations is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{142a}{33g}\right)}.$$